

# Digital Signal Processing

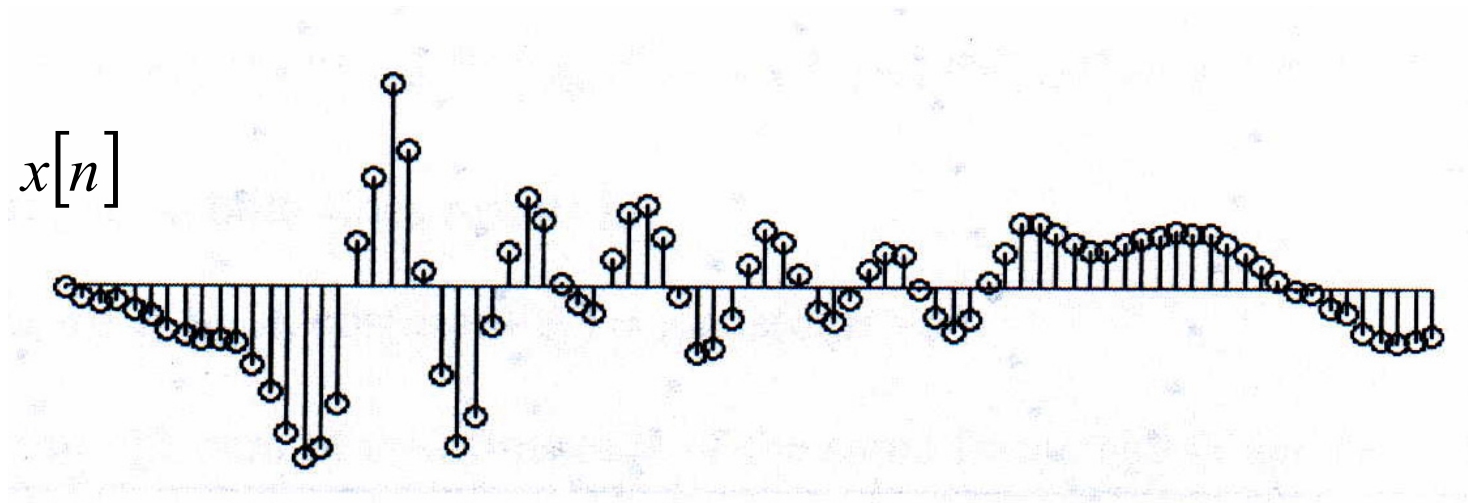
Berlin Chen 2004

## References:

1. A. V. Oppenheim and R. W. Schaffer, *Discrete-time Signal Processing*, 1999
2. X. Huang et. al., *Spoken Language Processing*, Chapters 5, 6
3. J. R. Deller et. al., *Discrete-Time Processing of Speech Signals*, Chapters 4-6
4. J. W. Picone, "Signal modeling techniques in speech recognition," *proceedings of the IEEE*, September 1993, pp. 1215-1247

# Digital Signal Processing

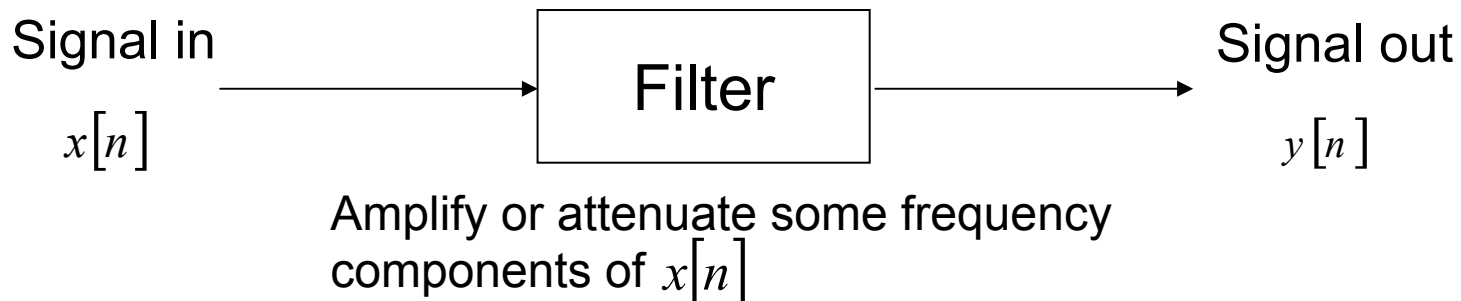
- Digital Signal
  - Discrete-time signal with discrete amplitude



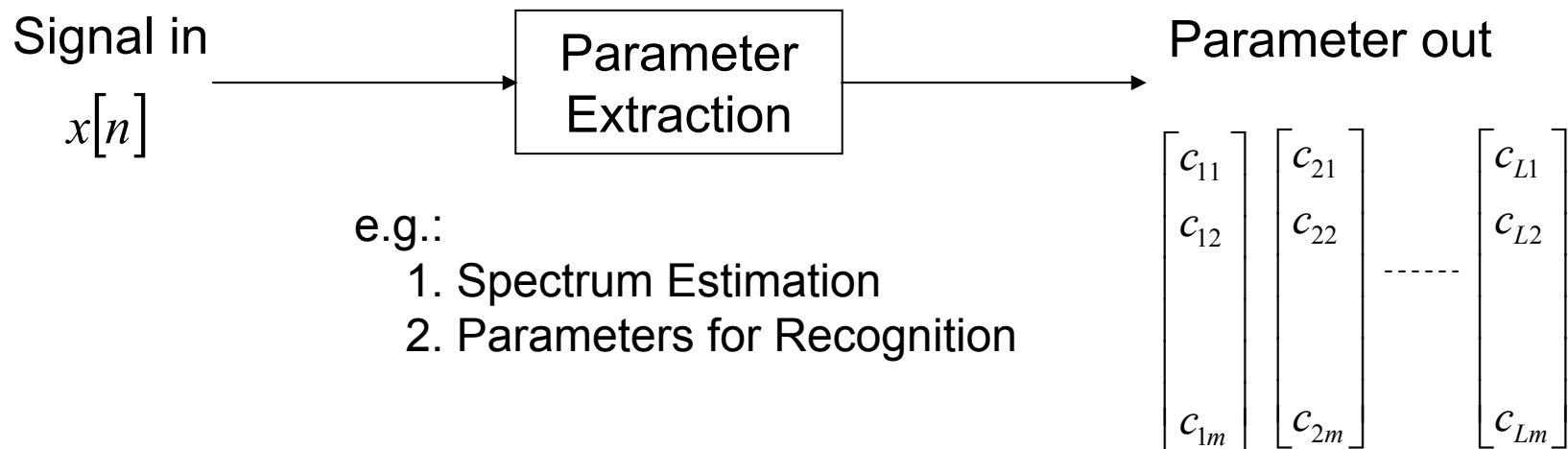
- Digital Signal Processing
  - Manipulate digital signals in a digital computer

# Two Main Approaches to Digital Signal Processing

- Filtering



- Parameter Extraction



# Sinusoid Signals

$$x[n] = A \cos(\omega n + \phi)$$

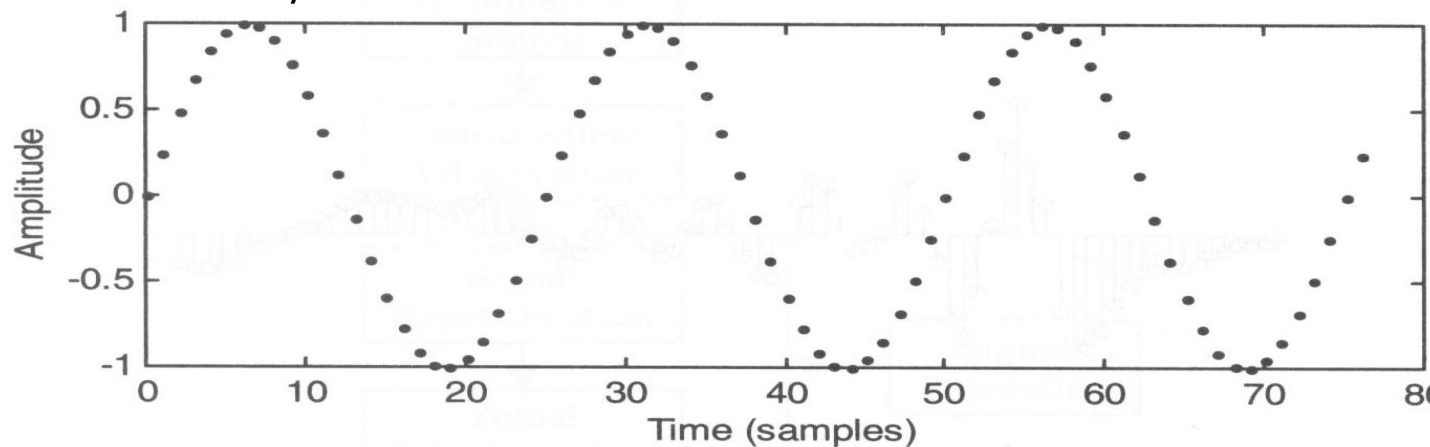
$f$  : normalized frequency  
 $0 \leq f \leq 1$

–  $A$  : amplitude (振幅)

–  $\omega$  : angular frequency (角频率),  $\omega = 2\pi f = \frac{2\pi}{T}$

–  $\phi$  : phase (相角)

Period, represented  
by number of samples



$$x[n] = A \cos\left(\omega n - \frac{\pi}{2}\right)$$

$T = 25$  samples

$$x[n] = x[n + N] = A \cos\left(\omega(n + N) - \frac{\pi}{2}\right)$$

$$\Rightarrow \omega N = 2\pi k \Rightarrow \omega = \frac{2\pi k}{N} (k = 1, 2, \dots)$$

- E.g., speech signals can be decomposed as sums of sinusoids

## Sinusoid Signals (cont.)

- $x[n]$  is periodic with a period of  $N$  (samples)
  - ⇒  $x[n + N] = x[n]$
  - ⇒  $A \cos(\omega(n + N) + \phi) = A \cos(\omega n + \phi)$
  - ⇒  $\omega N = 2\pi k \quad (k = 1, 2, \dots)$
  - ⇒  $\omega = \frac{2\pi}{N}$
- Examples (sinusoid signals)
  - $x_1[n] = \cos(\pi n / 4)$  is periodic with period  $N=8$
  - $x_2[n] = \cos(3\pi n / 8)$  is periodic with period  $N=16$
  - $x_3[n] = \cos(n)$  is not periodic

# Sinusoid Signals (cont.)

$$\begin{aligned}x_1[n] &= \cos(\pi n / 4) \\ &= \cos\left(\frac{\pi}{4}n\right) = \cos\left(\frac{\pi}{4}(n + N_1)\right) = \cos\left(\frac{\pi}{4}n + \frac{\pi}{4}N_1\right) \\ &\Rightarrow \frac{\pi}{4}N_1 = 2\pi \cdot k \Rightarrow 8 \cdot k \quad (N_1 \text{ and } k \text{ are positive integers})\end{aligned}$$

$$\therefore N_1 = 8$$

$$\begin{aligned}x_2[n] &= \cos(3\pi n / 8) \\ &= \cos\left(\frac{3\pi}{8} \cdot n\right) = \cos\left(\frac{3\pi}{8}(n + N_2)\right) = \cos\left(\frac{3\pi}{8} \cdot n + \frac{3\pi}{8} \cdot N_2\right) \\ &\Rightarrow \frac{3\pi}{8} \cdot N_2 = 2\pi \cdot k \Rightarrow N_2 = \frac{16}{3}k \quad (N_2 \text{ and } k \text{ are positive numbers})\end{aligned}$$

$$\therefore N_2 = 16$$

$$\begin{aligned}x_3[n] &= \cos(n) \\ &= \cos(1 \cdot n) = \cos(1 \cdot (n + N_3)) = \cos(n + N_3) \\ &\Rightarrow N_3 = 2\pi \cdot k\end{aligned}$$

$\therefore N_3$  and  $k$  are positive integers

$\therefore N_3$  doesn't exist !

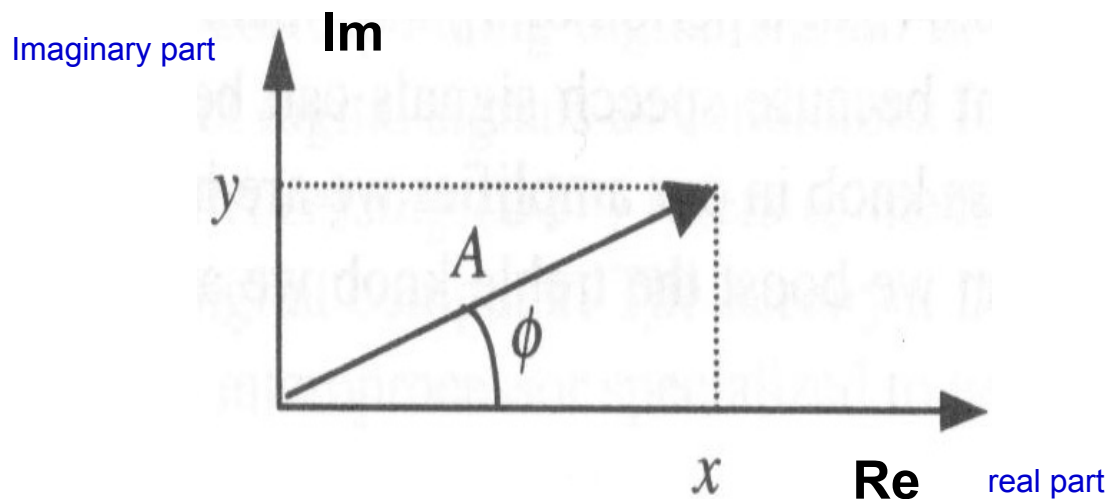
# Sinusoid Signals (cont.)

- Complex Exponential Signal
  - Use **Euler's relation** to express complex numbers

$$z = x + jy$$

$$\Rightarrow z = Ae^{j\phi} = A \left( \underbrace{\cos \phi}_{\frac{x}{\sqrt{x^2 + y^2}}} + j \underbrace{\sin \phi}_{\frac{y}{\sqrt{x^2 + y^2}}} \right)$$

( $A$  is a real number)



$$x = A \cos \phi$$

$$y = A \sin \phi$$

# Sinusoid Signals (cont.)

- A Sinusoid Signal

$$\begin{aligned}x[n] &= A \cos(\omega n + \phi) \\ &= \text{Re} \left\{ A e^{j(\omega n + \phi)} \right\} \\ &= \text{Re} \left\{ A e^{j\omega n} e^{j\phi} \right\}\end{aligned}$$

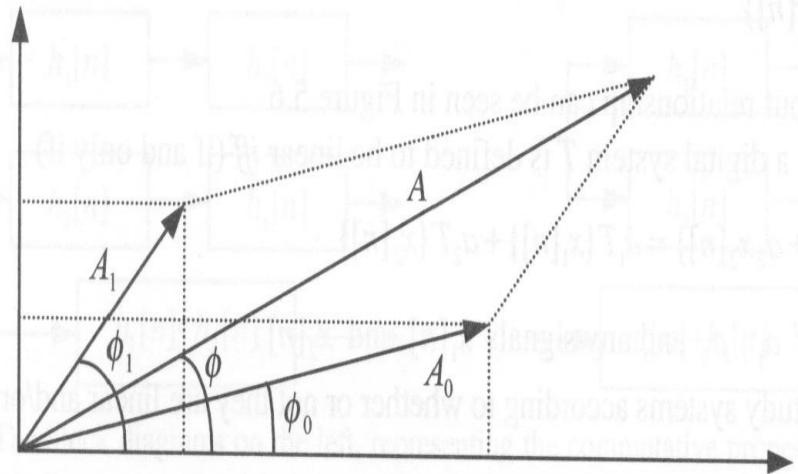
real part



## Sinusoid Signals (cont.)

- Sum of two complex exponential signals with same frequency

$$\begin{aligned} & A_0 e^{j(\omega n + \phi_0)} + A_1 e^{j(\omega n + \phi_1)} \\ &= e^{j\omega n} (A_0 e^{j\phi_0} + A_1 e^{j\phi_1}) \\ &= e^{j\omega n} A e^{j\phi} \\ &= A e^{j(\omega n + \phi)} \end{aligned}$$



$A$ ,  $A_0$  and  $A_1$  are real numbers

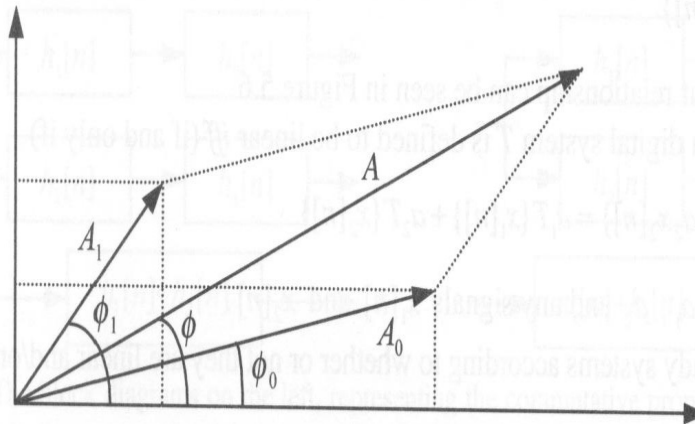
- When only the real part is considered

$$A_0 \cos(\omega n + \phi_0) + A_1 \cos(\omega n + \phi_1) = A \cos(\omega n + \phi)$$

- The sum of  $N$  sinusoids of the same frequency is another sinusoid of the same frequency

# Sinusoid Signals (cont.)

- Trigonometric Identities



$$\tan \phi = \frac{A_0 \sin \phi_0 + A_1 \sin \phi_1}{A_0 \cos \phi_0 + A_1 \cos \phi_1}$$

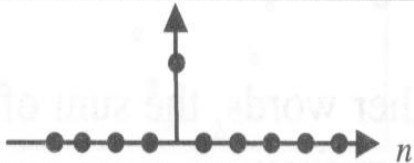
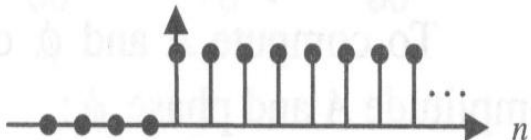
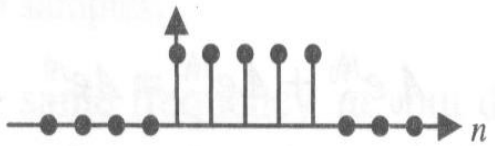
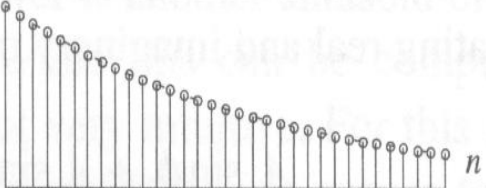
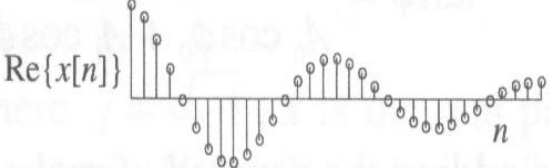
$$A^2 = (A_0 \cos \phi_0 + A_1 \cos \phi_1)^2 + (A_0 \sin \phi_0 + A_1 \sin \phi_1)^2$$

$$A^2 = A_0^2 + A_1^2 + 2 A_0 A_1 (\cos \phi_0 \cos \phi_1 + \sin \phi_0 \sin \phi_1)$$

$$= A_0^2 + A_1^2 + 2 A_0 A_1 \cos (\phi_0 - \phi_1)$$

# Some Digital Signals

**Table 5.1** Some useful digital signals: the Kronecker delta, unit step, rectangular signal, real exponential ( $a < 1$ ) and real part of a complex exponential ( $r < 1$ ).

<p><i>Kronecker delta, or unit impulse</i></p>	$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$	
<p><i>Unit step</i></p>	$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$	
<p><i>Rectangular signal</i></p>	$\text{rect}_N[n] = \begin{cases} 1 & 0 \leq n < N \\ 0 & \text{otherwise} \end{cases}$	
<p><i>Real exponential</i></p>	$x[n] = a^n u[n]$	
<p><i>Complex exponential</i></p>	$x[n] = a^n u[n] = r^n e^{jn\omega_0} u[n]$ $= r^n (\cos n\omega_0 + j \sin n\omega_0) u[n]$	

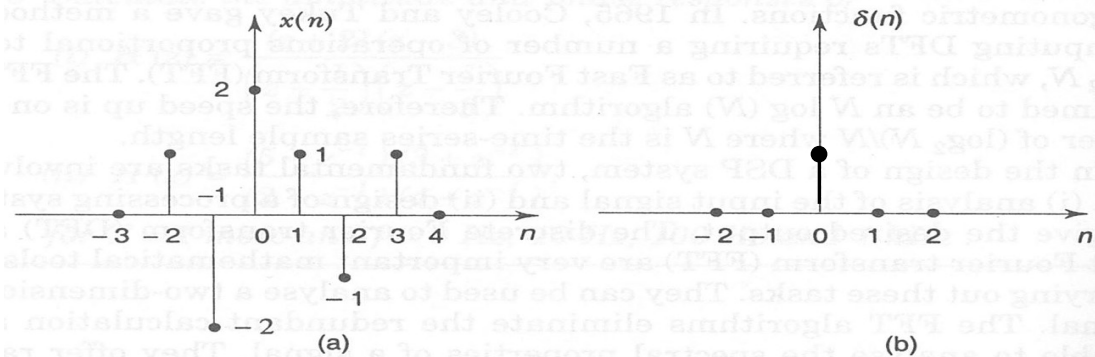
# Some Digital Signals

- Any signal sequence  $x[n]$  can be represented as a sum of **shift** and **scaled** unit impulse sequences (signals)

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k]$$

$n = \dots, -2, -1, 0, 1, 2, 3, \dots$

$x[k]$  scale/weighted  
 $\delta[n-k]$  Time-shifted unit impulse sequence

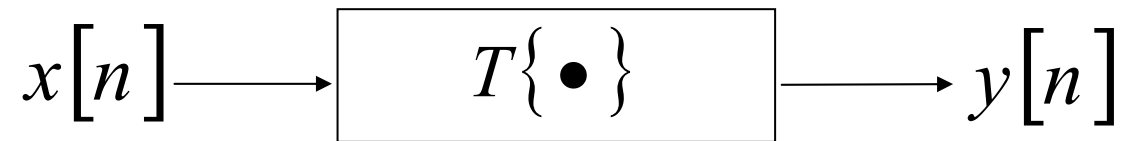


$$\begin{aligned}
 x[n] &= \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] = \sum_{k=-2}^3 x[k] \delta[n-k] \\
 &= x[-2] \delta[n+2] + x[-1] \delta[n+1] + x[0] \delta[n] + x[1] \delta[n-1] + x[2] \delta[n-2] + x[3] \delta[n-3] \\
 &= (1) \delta[n+2] + (-2) \delta[n+1] + (2) \delta[n] + (1) \delta[n-1] + (-1) \delta[n-2] + (1) \delta[n-3]
 \end{aligned}$$

# Digital Systems

- A digital system  $T$  is a system that, given an input signal  $x[n]$ , generates an output signal  $y[n]$

$$y[n] = T \{ x[n] \}$$



# Properties of Digital Systems

- Linear

- Linear combination of inputs maps to linear combination of outputs

$$T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\}$$

- Time-invariant (Time-shift)

- A time shift of in the input by  $m$  samples give a shift in the output by  $m$  samples

$$y[n \pm m] = T\{x[n \pm m]\}, \quad \forall m$$

time shift

$x[n - m]$  (if  $m > 0$ )  $\Rightarrow$  right shift  $m$  samples  
 $x[n + m]$  (if  $m > 0$ )  $\Rightarrow$  left shift  $m$  samples

# Properties of Digital Systems (cont.)

- Linear time-invariant (LTI)
  - The system output can be expressed as a convolution (迴旋積分) of the input  $x[n]$  and the *impulse response*  $h[n]$
  - The system can be characterized by the system's *impulse response*  $h[n]$ , which also is a signal sequence
    - If the input  $x[n]$  is impulse  $\delta[n]$ , the output is  $h[n]$



# Properties of Digital Systems (cont.)

- Linear time-invariant (LTI)

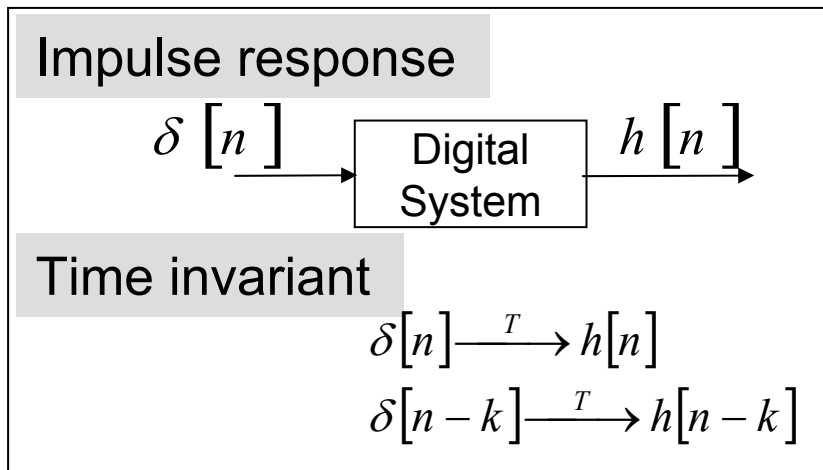
– Explanation:

$$x[n] = \sum_{k=-\infty}^{\infty} \underbrace{x[k]}_{\text{scale}} \underbrace{\delta[n-k]}_{\text{Time-shifted unit impulse sequence}}$$

$$\begin{aligned} \Rightarrow T\{x[n]\} &= T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\} \\ &= \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= x[n]*h[n] \end{aligned}$$

*linear*

*Time-invariant*

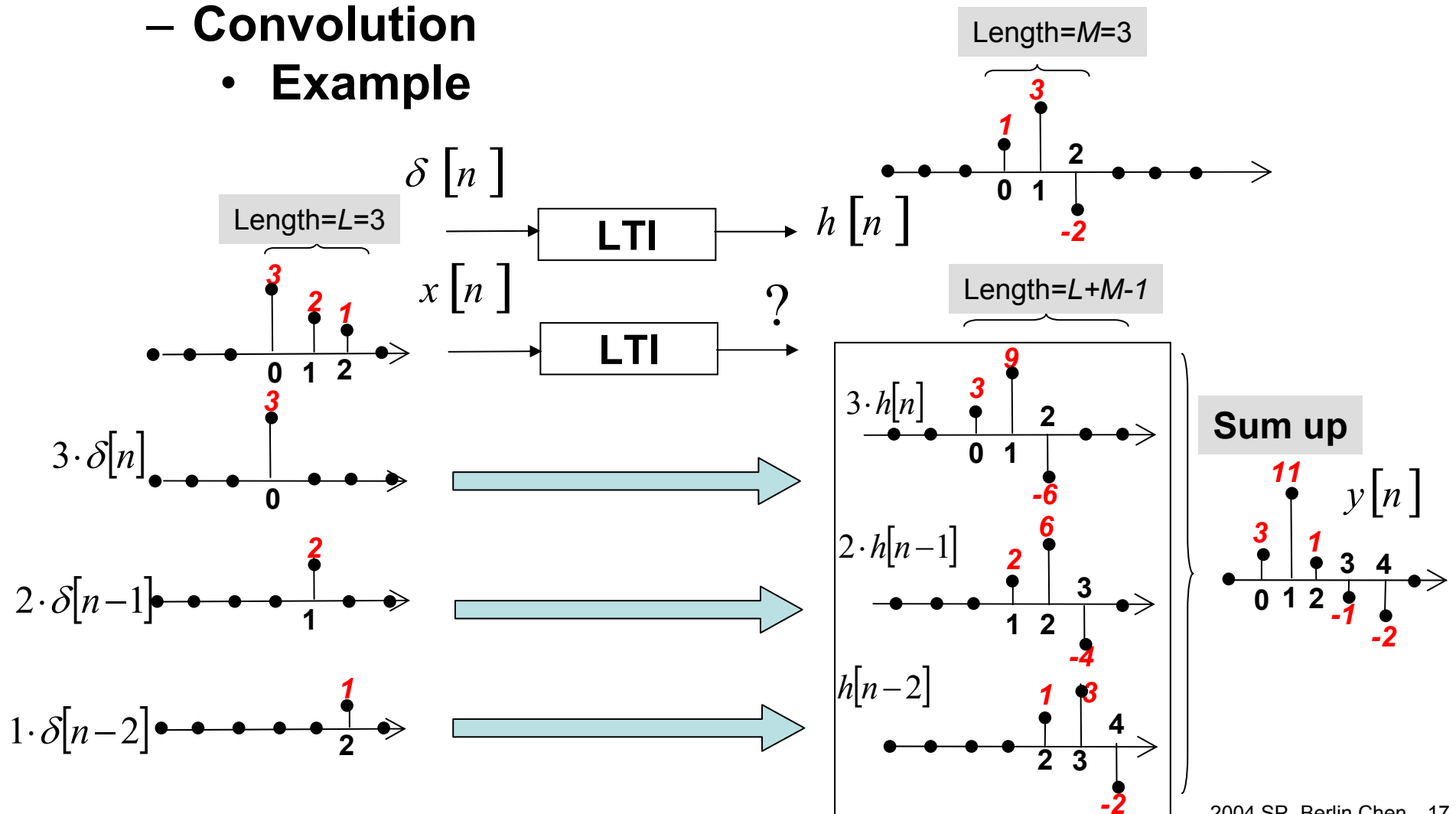


**convolution**



# Properties of Digital Systems (cont.)

- Linear time-invariant (LTI)
  - Convolution
    - Example



# Properties of Digital Systems (cont.)

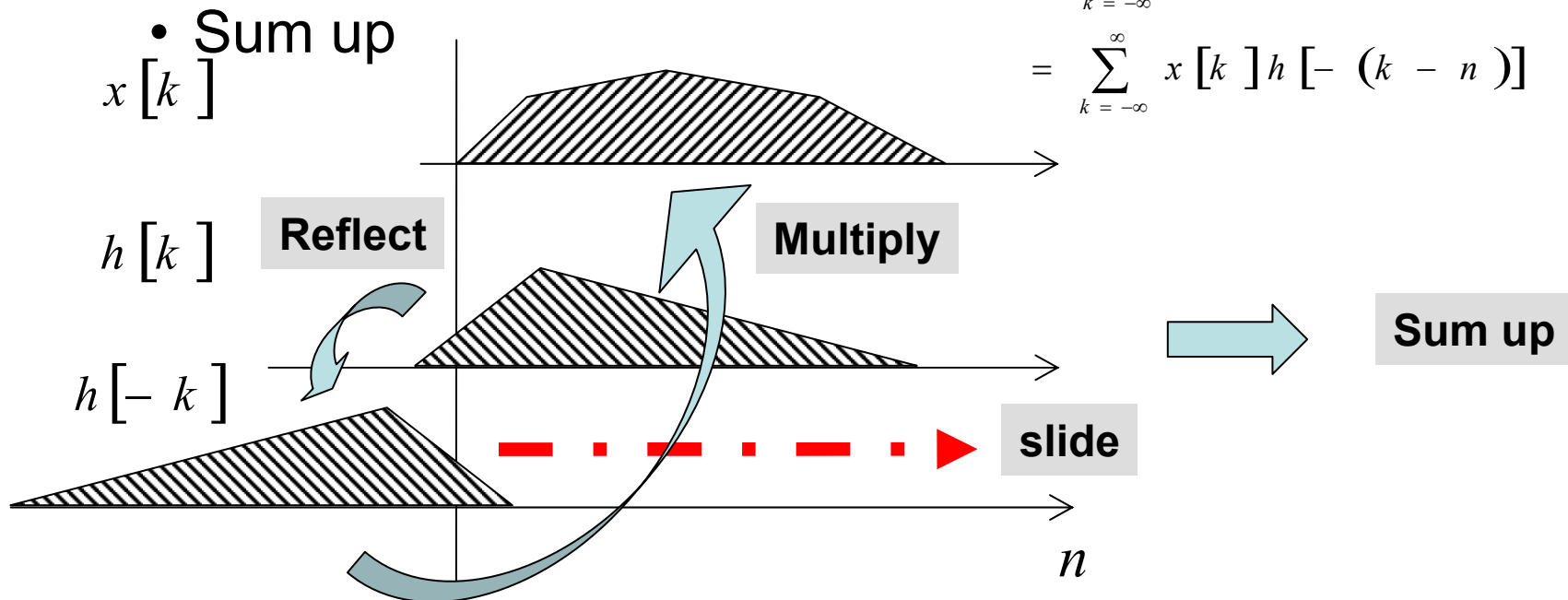
- Linear time-invariant (LTI)

- Convolution: **Generalization**

- Reflect  $h[k]$  about the origin ( $\rightarrow h[-k]$ )
- Slide ( $h[-k] \rightarrow h[-k+n]$  or  $h[-(k-n)]$ ), multiply it with  $x[k]$

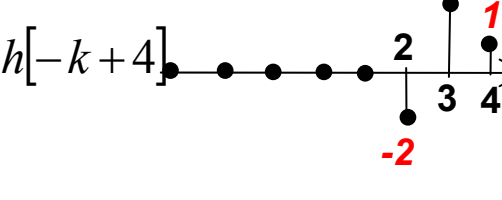
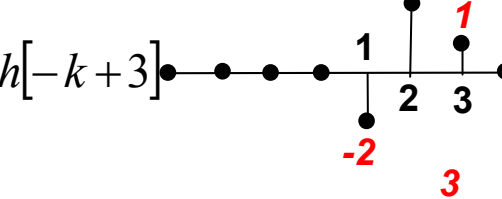
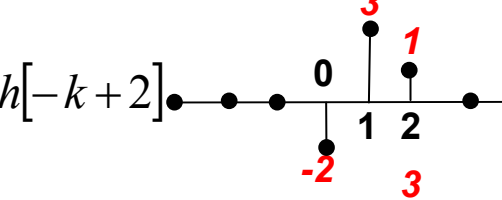
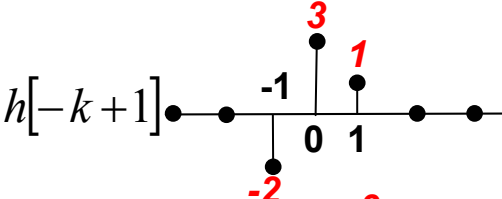
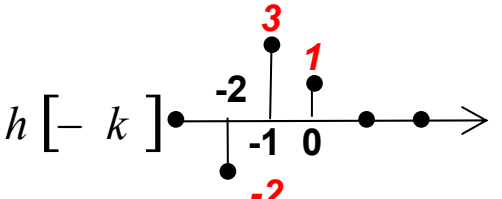
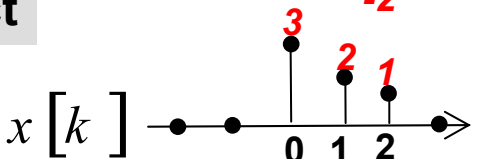
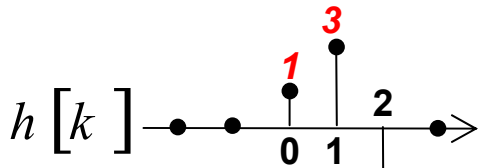
$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x[k] h[-(k-n)]$$



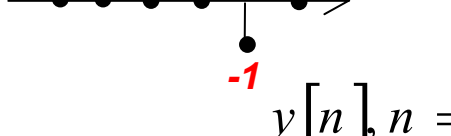
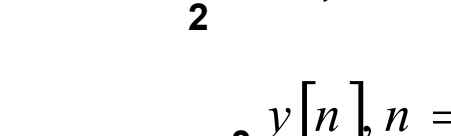
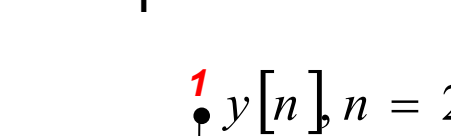
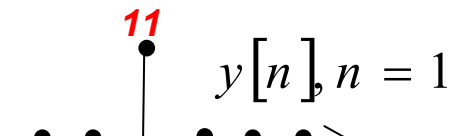
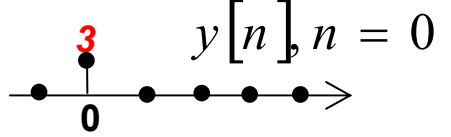
# Convolution

Reflect

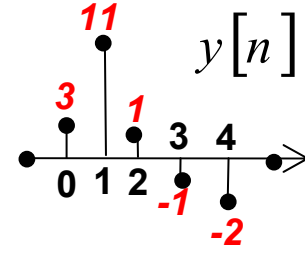


$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

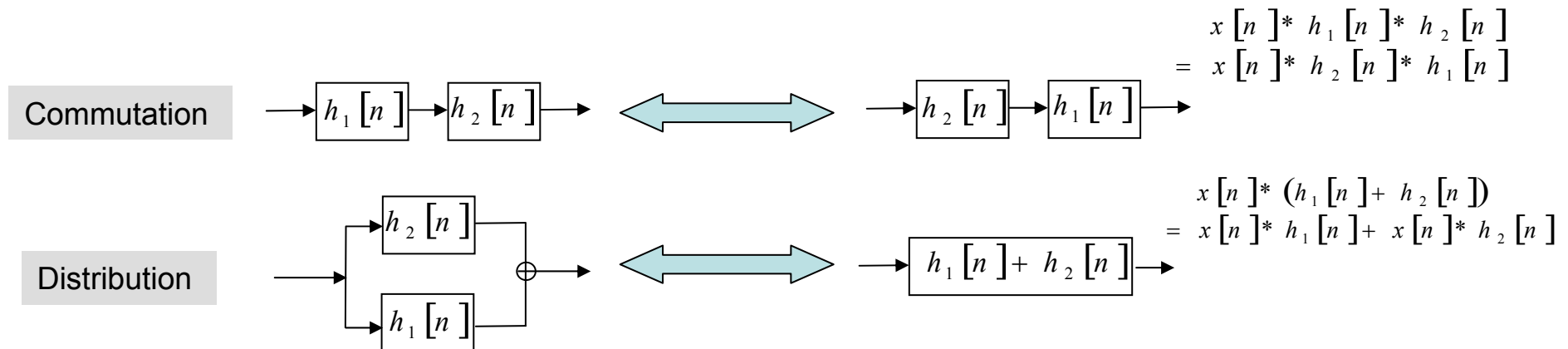


Sum up



# Properties of Digital Systems (cont.)

- Linear time-invariant (LTI)
  - Convolution is commutative and distributive



- An impulse response has finite duration
  - » **Finite-Impulse Response (FIR)**
- An impulse response has infinite duration
  - » **Infinite-Impulse Response (IIR)**

$$\begin{aligned}
 y[n] &= x[n] * h[n] \\
 &= h[n] * x[n] \\
 &= \sum_{k=-\infty}^{\infty} x[k] h[n-k] \\
 &= \sum_{k=-\infty}^{\infty} h[k] x[n-k]
 \end{aligned}$$

# Properties of Digital Systems (cont.)

- Prove convolution is commutative

$$\begin{aligned}y[n] &= x[n] * h[n] \\&= \sum_{k=-\infty}^{\infty} x[k] h[n-k] \\&= \sum_{m=-\infty}^{\infty} x[n-m] h[m] \quad (\text{let } m = n - k) \\&= h[n] * x[n]\end{aligned}$$

# Properties of Digital Systems (cont.)

- Linear time-varying System
  - E.g.,  $y[n] = x[n] \cos \omega_0 n$  is an amplitude modulator

$$\text{suppose } x_1[n] = x[n - n_0] \Rightarrow y_1[n] \stackrel{?}{=} y[n - n_0]$$

$$y_1[n] = x_1[n] \cos \omega_0 n = x[n - n_0] \cos \omega_0 n$$

$$\text{But } y[n - n_0] = x[n - n_0] \cos \omega_0 (n - n_0)$$

# Properties of Digital Systems (cont.)

- **Bounded Input and Bounded Output (BIBO):** stable

$$|x[n]| \leq B_x < \infty \quad \forall n$$

$$|y[n]| \leq B_y < \infty \quad \forall n$$

- **A LTI system** is BIBO if only if  $h[n]$  is absolutely summable

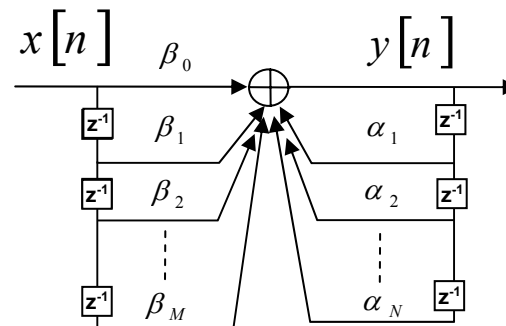
$$\sum_{k=-\infty}^{\infty} |h[k]| \leq \infty$$

# Properties of Digital Systems (cont.)

- **Causality**

- A system is “casual” if for every choice of  $n_0$ , the output sequence value at indexing  $n=n_0$  depends on only the input sequence value for  $n \leq n_0$

$$y[n_0] = \sum_{k=1}^K \alpha_k y[n_0 - k] + \sum_{k=m}^M \beta_k x[n_0 - m]$$

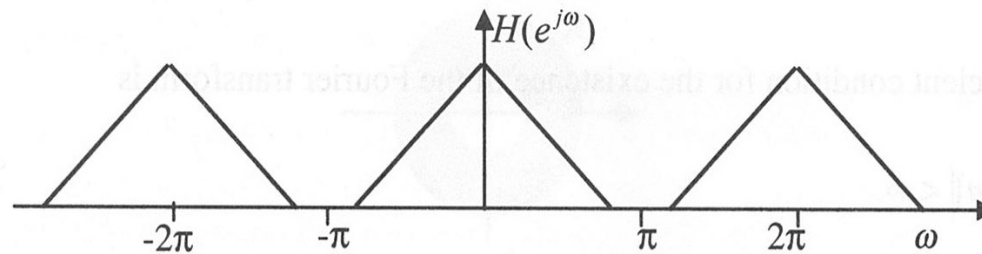


- Any noncausal FIR can be made causal by adding sufficient long delay



# Discrete-Time Fourier Transform (DTFT)

- Frequency Response  $H(e^{j\omega})$ 
  - Defined as the discrete-time Fourier Transform of  $h[n]$
  - $H(e^{j\omega})$  is continuous and is periodic with period =  $2\pi$



proportional to two times of the sampling frequency

Figure 5.8  $H(e^{j\omega})$  is a periodic function of  $\omega$ .

- $H(e^{j\omega})$  is a complex function of  $\omega$

$$\begin{aligned}
 H(e^{j\omega}) &= H_r(e^{j\omega}) + jH_i(e^{j\omega}) \\
 &= \underbrace{|H(e^{j\omega})|}_{\text{magnitude}} e^{j\underbrace{\angle H(e^{j\omega})}_{\text{phase}}}
 \end{aligned}$$

# Discrete-Time Fourier Transform (cont.)

- Representation of Sequences by Fourier Transform

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} \quad \text{DTFT}$$

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \quad \text{Inverse DTFT}$$

- A sufficient condition for the existence of Fourier transform

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \quad \text{absolutely summable}$$

Fourier transform is invertible:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

$$\begin{aligned} h[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{m=-\infty}^{\infty} h[m] e^{-j\omega m} e^{j\omega n} d\omega \\ &= \sum_{m=-\infty}^{\infty} h[m] \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega(n-m)} d\omega = \sum_{m=-\infty}^{\infty} h[m] \delta[n-m] = h[n] \end{aligned}$$

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega(n-m)} d\omega \\ &= \frac{1}{j2\pi(n-m)} \left[ e^{j\omega(n-m)} \right]_{-\pi}^{\pi} \\ &= \frac{\sin \pi(n-m)}{\pi(n-m)} \\ &= \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \\ &= \delta[n-m] \end{aligned}$$

# Discrete-Time Fourier Transform (cont.)

- Convolution Property

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k] h[n-k] e^{-j\omega n}$$

$$\begin{aligned} n' &= n - k \\ \Rightarrow n &= n' + k \\ \Rightarrow -n &= -n' - k \end{aligned}$$

$$\begin{aligned} &= \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k} \left( \sum_{n'=-\infty}^{\infty} h[n'] e^{-j\omega n'} \right) \\ &= X(e^{j\omega}) H(e^{j\omega}) \end{aligned}$$

$$\therefore x[n] * h[n] \Leftrightarrow X(e^{j\omega}) H(e^{j\omega})$$

$$\begin{aligned} Y(e^{j\omega}) &= X(e^{j\omega}) H(e^{j\omega}) \\ \Rightarrow |Y(e^{j\omega})| &= |X(e^{j\omega})| |H(e^{j\omega})| \\ \Rightarrow \angle Y(e^{j\omega}) &= \angle X(e^{j\omega}) + \angle H(e^{j\omega}) \end{aligned}$$

# Discrete-Time Fourier Transform (cont.)

\*: complex conjugate

$$z = x + jy \Rightarrow z^* = x - jy$$

- Parseval's Theorem

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \boxed{\text{power spectrum}} \left| X(e^{j\omega}) \right|^2 d\omega$$

The total energy of a signal can be given in either the time or frequency domain.

– Define the autocorrelation of signal  $x[n]$

$$R_{xx}[n] = \sum_{m=-\infty}^{\infty} x[m+n]x^*[m]$$

$$l = m + n$$

$$\Rightarrow m = l - n = -(n - l)$$

$$= \sum_{l=-\infty}^{\infty} x[l]x^*[-(n-l)] = x[n] * x^*[-n]$$

$\Leftrightarrow$

$$S_{xx}(\omega) = X(\omega)X^*(\omega) = |X(\omega)|^2$$

$$R_{xx}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{xx}(\omega) e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 e^{j\omega n} d\omega$$

Set  $n = 0$

$$R_{xx}[0] = \sum_{m=-\infty}^{\infty} x[m]x^*[m] = \sum_{m=-\infty}^{\infty} |x[m]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

# Discrete-Time Fourier Transform (DTFT)

- A LTI system with impulse response  $h[n]$ 
  - What is the output  $y[n]$  for  $x[n] = A \cos(\omega n + \phi)$

$$x_1[n] = jA \sin(\omega n + \phi)$$

$$x_0[n] = x[n] + x_1[n] = Ae^{j(\omega n + \phi)}$$

$$y_0[n] = Ae^{j(\omega n + \phi)} * h[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k] Ae^{j(\omega(n-k) + \phi)}$$

$$= Ae^{j(\omega n + \phi)} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

$$= Ae^{j(\omega n + \phi)} H(e^{j\omega})$$

$$= Ae^{j(\omega n + \phi)} |H(e^{j\omega})| e^{j\angle H(e^{j\omega})}$$

$$= A |H(e^{j\omega})| e^{j(\omega n + \phi) + j\angle H(e^{j\omega})}$$

$$= A |H(e^{j\omega})| \cos((\omega n + \phi) + \angle H(e^{j\omega})) + jA |H(e^{j\omega})| \sin((\omega n + \phi) + \angle H(e^{j\omega}))$$

$$\Rightarrow y[n] = A |H(e^{j\omega})| \cos((\omega n + \phi) + \angle H(e^{j\omega}))$$

$$|H(e^{j\omega})| > 1 \quad \text{amplify}$$

$$|H(e^{j\omega})| < 1 \quad \text{attenuate}$$

System's frequency response

$y[n]$

$y_1[n]$

# Discrete-Time Fourier Transform (cont.)

Property	Signal	Fourier Transform
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(e^{j\omega}) + bX_2(e^{j\omega})$
Symmetry	$x[-n]$	$X(e^{-j\omega})$
	$x^*[n]$	$X^*(e^{-j\omega})$
	$x^*[-n]$	$X^*(e^{j\omega})$
	$x[n]$ real	$X(e^{j\omega})$ is Hermitian $X(e^{-j\omega}) = X^*(e^{j\omega})$ $ X(e^{j\omega}) $ is even <sup>6</sup> $\text{Re}\{X(e^{j\omega})\}$ is even $\text{arg}\{X(e^{j\omega})\}$ is odd <sup>7</sup> $\text{Im}\{X(e^{j\omega})\}$ is odd
	Even $\{x[n]\}$	$\text{Re}\{X(e^{j\omega})\}$
	Odd $\{x[n]\}$	$j \text{Im}\{X(e^{j\omega})\}$
Time-shifting	$x[n - n_0]$	$X(e^{j\omega})e^{-j\omega n_0}$
Modulation	$x[n]e^{j\omega_0 n}$	$X(e^{j(\omega - \omega_0)})$
	$x[n]z_0^n$	
Convolution	$x[n] * h[n]$	$X(e^{j\omega})H(e^{j\omega})$
	$x[n]y[n]$	$\frac{1}{2\pi} X(e^{j\omega}) * Y(e^{j\omega})$
Parseval's Theorem	$R_{xx}[n] = \sum_{m=-\infty}^{\infty} x[m+n]x^*[m]$	$S_{xx}(\omega) =  X(\omega) ^2$

# Z-Transform

- z-transform is a generalization of (Discrete-Time) Fourier transform

$$h[n] \longrightarrow H(e^{j\omega})$$

$$h[n] \longrightarrow H(z)$$

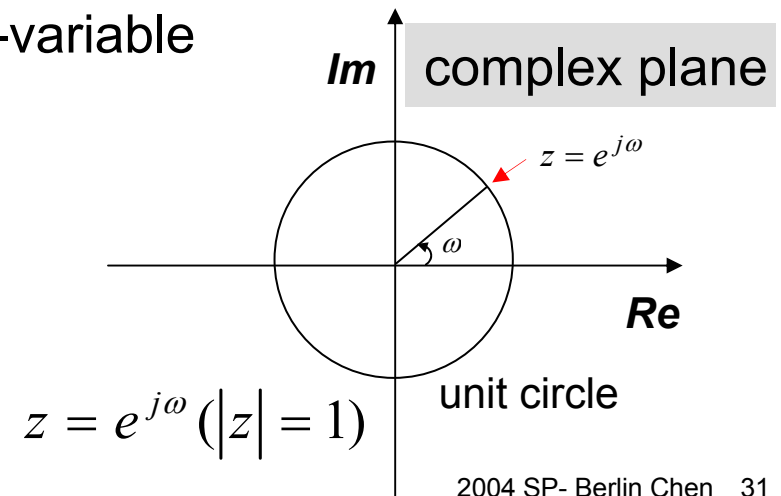
- z-transform of  $h[n]$  is defined as

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

- Where  $z = re^{j\omega}$ , a complex-variable
- For Fourier transform

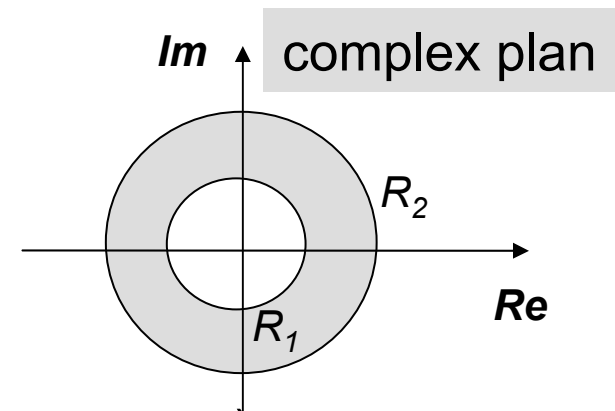
$$H(e^{j\omega}) = H(z) \Big|_{z=e^{j\omega}}$$

- z-transform evaluated on the unit circle



# Z-Transform (cont.)

- Fourier transform vs. z-transform
  - Fourier transform used to plot the frequency response of a filter
  - z-transform used to analyze more general filter characteristics, e.g. stability



- ROC (Region of Converge)
  - Is the set of  $z$  for which z-transform exists (converges)

$$\sum_{n=-\infty}^{\infty} |h[n]| |z|^{-n} < \infty \quad \text{absolutely summable}$$

- In general, ROC is a **ring-shaped region** and the Fourier transform exists if ROC includes the unit circle ( $|z|=1$ )



## Z-Transform (cont.)

$$\begin{aligned}y[n] &= x[n] * h[n] \\ &= h[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} x[k] h[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k] x[n-k]\end{aligned}$$

- An LTI system is defined to be **causal**, if its impulse response is a causal signal, i.e.

$$h[n] = 0 \quad \text{for } n < 0 \quad \text{Right-sided sequence}$$

- Similarly, **anti-causal** can be defined as

$$h[n] = 0 \quad \text{for } n > 0 \quad \text{Left-sided sequence}$$

- An LTI system is defined to be **stable**, if for every bounded input it produces a bounded output

- Necessary condition:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- That is Fourier transform exists, and therefore z-transform include the unit circle in its region of converge

# Z-Transform (cont.)

- **Right-Sided Sequence**

- E.g., the exponential signal

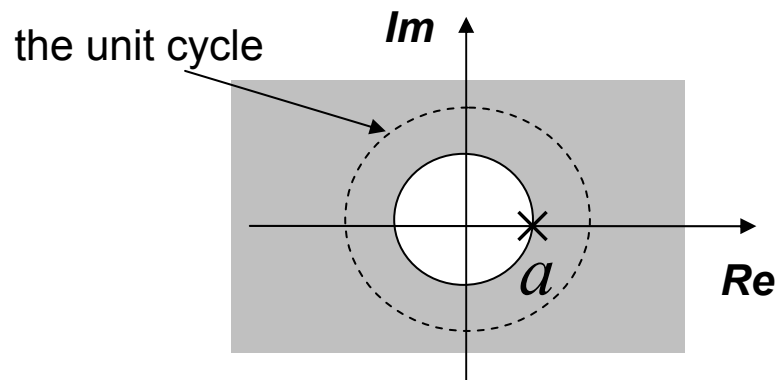


$$1. \quad h_1[n] = a^n u[n], \quad \text{where } u[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$H_1(z) = \sum_{n=-\infty}^{\infty} a^n z^{-n} = \sum_{n=-\infty}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} \quad \begin{array}{l} \text{have a pole at } Z = a \\ \text{(Pole: z-transform goes to infinity)} \end{array}$$

$\text{If } |az^{-1}| < 1$

$$\therefore \text{ROC}_1 \text{ is } |z| > |a|$$



Fourier transform of  $h_1[n]$  exists if  $|a| < 1$

# Z-Transform (cont.)

- **Left-Sided Sequence**

- E.g.

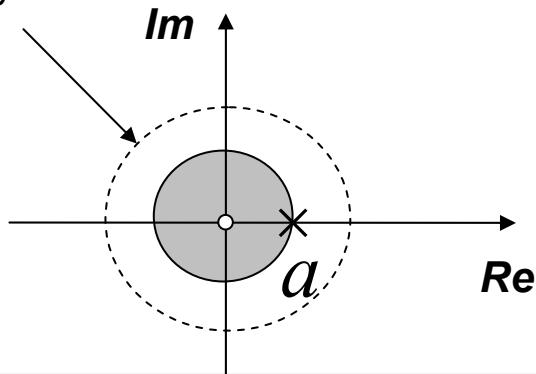
2.  $h_2[n] = -a^n u[-n-1]$  ( $n=0, -1, -2, \dots, -\infty$ )

$$H_2(z) = - \sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = - \sum_{n=-\infty}^{-1} a^n z^{-n} \quad \text{If } |a^{-1}z| < 1$$

$$= - \sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = 1 - \frac{1}{1 - a^{-1}z} = \frac{-a^{-1}z}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}}$$

$\therefore ROC_2$  is  $|z| < |a|$

the unit cycle



when  $|a| < 1$ , the Fourier transform of  $h_2[n]$  doesn't exist, because  $h_2[n]$  will go exponentially as  $n \rightarrow -\infty$

# Z-Transform (cont.)

- **Two-Sided Sequence**



– E.g.

$$3. \quad h_3[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

$$\left(-\frac{1}{3}\right)^n u[n] \quad \xrightarrow{z} \quad \frac{1}{1 + \frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$$

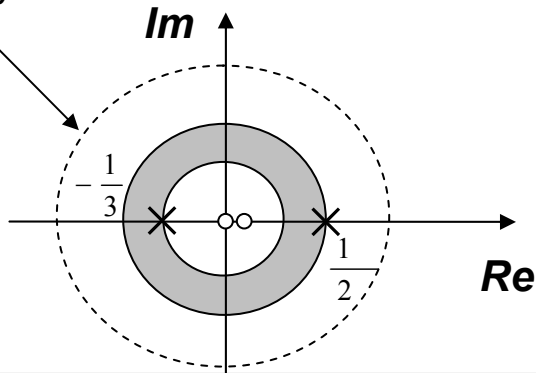
$$-\left(\frac{1}{2}\right)^n u[-n-1] \quad \xrightarrow{z} \quad \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| < \frac{1}{2}$$



$$H_3(z) = \frac{1}{1 + \frac{1}{3}z^{-1}} + \frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{2z\left(z - \frac{1}{12}\right)}{\left(z + \frac{1}{3}\right)\left(z - \frac{1}{2}\right)}$$

$\therefore ROC_3$  is  $|z| < \frac{1}{2}$  and  $|z| > \frac{1}{3}$

the unit cycle



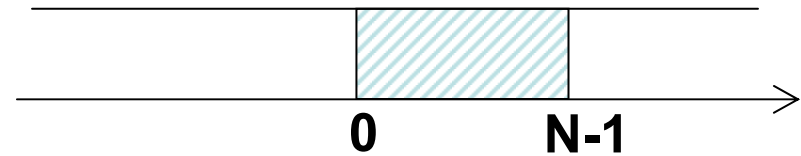
Fourier transform of  $h_3[n]$  doesn't exist, because  $ROC_3$  doesn't include the unit circle

# Z-Transform (cont.)

- **Finite-length Sequence**

- E.g.

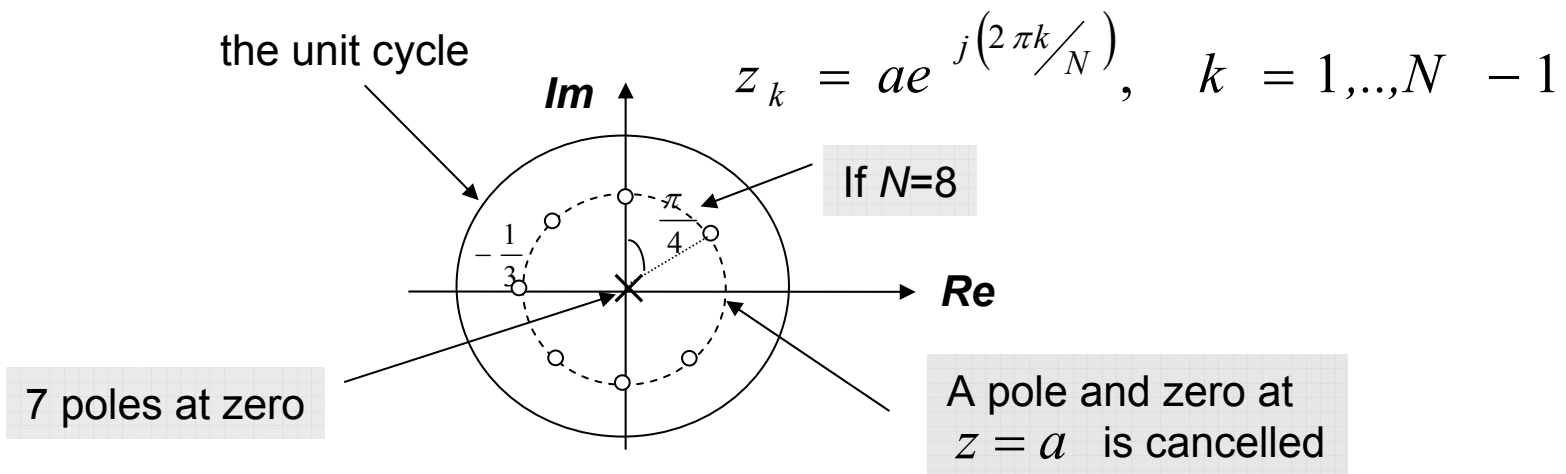
$$3. \quad h_4[n] = \begin{cases} a^n, & 0 \leq n \leq N-1 \\ 0, & \text{others} \end{cases}$$



$$z^{N-1} + az^{N-2} + a^2z^{N-3} + \dots + a^{N-1}$$

$$H_4(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n = \frac{1 - (az^{-1})^N}{1 - az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}$$

$\therefore ROC_4$  is entire  $z$ -plane except  $z = 0$



# Z-Transform (cont.)

- **Properties of z-transform**

1. If  $h[n]$  is right-sided sequence, i.e.  $h[n] = 0, n \leq n_1$  and if *ROC* is **the exterior of some circle**, the **all finite**  $z$  for which  $|z| > r_0$  will be in *ROC*

- If  $n_1 \geq 0$ , *ROC* will include  $z = \infty$

A causal sequence is right-sided with  $n_1 \geq 0$   
 $\therefore$  *ROC* is the exterior of circle including  $z = \infty$

2. If  $h[n]$  is left-sided sequence, i.e.  $h[n] = 0, n \geq n_2$ , the *ROC* is **the interior of some circle**,

- If  $n_2 < 0$ , *ROC* will include  $z = 0$

3. If  $h[n]$  is two-sided sequence, the *ROC* is a **ring**

4. The *ROC* can't contain any poles

# Summary of the Fourier and z-transforms

Table 5.5 Properties of the Fourier and z-transforms.

Property	Signal	Fourier Transform	z-Transform
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(e^{j\omega}) + bX_2(e^{j\omega})$	$aX_1(z) + bX_2(z)$
Symmetry	$x[-n]$	$X(e^{-j\omega})$	$X(z^{-1})$
	$x^*[n]$	$X^*(e^{-j\omega})$	$X^*(z^*)$
	$x^*[-n]$	$X^*(e^{j\omega})$	$X^*(1/z^*)$
	$x[n]$ real	$X(e^{j\omega})$ is Hermitian $X(e^{-j\omega}) = X^*(e^{j\omega})$ $ X(e^{j\omega}) $ is even <sup>6</sup> $\text{Re}\{X(e^{j\omega})\}$ is even $\text{arg}\{X(e^{j\omega})\}$ is odd <sup>7</sup> $\text{Im}\{X(e^{j\omega})\}$ is odd	$X(z^*) = X^*(z)$
	Even $\{x[n]\}$	$\text{Re}\{X(e^{j\omega})\}$	
	Odd $\{x[n]\}$	$j \text{Im}\{X(e^{j\omega})\}$	
Time-shifting	$x[n - n_0]$	$X(e^{j\omega})e^{-j\omega n_0}$	$X(z)z^{-n_0}$
Modulation	$x[n]e^{j\omega_0 n}$	$X(e^{j(\omega - \omega_0)})$	$X(e^{-j\omega_0} z)$
	$x[n]z_0^n$		$X(z/z_0)$
Convolution	$x[n] * h[n]$	$X(e^{j\omega})H(e^{j\omega})$	$X(z)H(z)$
	$x[n]y[n]$	$\frac{1}{2\pi} X(e^{j\omega}) * Y(e^{j\omega})$	
Parseval's Theorem	$R_{xx}[n] = \sum_{m=-\infty}^{\infty} x[m+n]x^*[m]$	$S_{xx}(\omega) =  X(\omega) ^2$	$X(z)X^*(1/z^*)$

# LTI Systems in the Frequency Domain

- **Example 1:** A complex exponential sequence  $x[n] = e^{j\omega n}$ 
  - System impulse response  $h[n]$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)}$$

$$= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

$$= H(e^{j\omega}) e^{j\omega n}$$

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= h[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} x[k] h[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k] x[n-k] \end{aligned}$$

$H(e^{j\omega})$ : the Fourier transform of the system impulse response. It is often referred to as the system frequency response.

- Therefore, a complex exponential input to an LTI system results in the same complex exponential at the output, but modified by  $H(e^{j\omega})$ 
  - The complex exponential is an eigenfunction of an LTI system, and  $H(e^{j\omega})$  is the associated eigenvalue

$$T\{x[n]\} = H(e^{j\omega}) x[n]$$



# LTI Systems in the Frequency Domain (cont.)

- **Example 2:** A sinusoidal sequence

$$x[n] = A \cos(\omega_0 n + \phi)$$

$$= \frac{A}{2} e^{j\phi} e^{j\omega_0 n} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 n}$$

$$x[n] = A \cos(\omega_0 n + \phi)$$

$$e^{j\theta} = \cos \theta + j \sin \theta$$

$$e^{-j\theta} = \cos \theta - j \sin \theta$$

$$\Rightarrow \cos \theta = \frac{1}{2} (e^{j\theta} + e^{-j\theta})$$

- System impulse response  $h[n]$

$$z = x + jy \quad \Rightarrow e^{j\omega} = \cos \omega + j \sin \omega$$

$$z^* = x - jy \quad \Rightarrow (e^{j\omega})^* = \cos \omega - j \sin \omega$$

$$e^{-j\omega} = \cos(-\omega) + j \sin(-\omega)$$

$$= \cos \omega - j \sin \omega$$

$$y[n] = H(e^{j\omega_0}) \frac{A}{2} e^{j\phi} e^{j\omega_0 n} + H(e^{-j\omega_0}) \frac{A}{2} e^{-j\phi} e^{-j\omega_0 n}$$

$$= \frac{A}{2} \left[ H(e^{j\omega_0}) e^{j(\omega_0 n + \phi)} + H^*(e^{j\omega_0}) e^{-j(\omega_0 n + \phi)} \right]$$

$$H(e^{-j\omega_0}) = H^*(e^{j\omega_0})$$

$$H^*(e^{j\omega_0}) = |H(e^{j\omega_0})| e^{-j\angle H(e^{j\omega_0})}$$

$$= \frac{A}{2} \left[ |H(e^{j\omega_0})| e^{j\angle H(e^{j\omega_0})} e^{j(\omega_0 n + \phi)} + |H(e^{j\omega_0})| e^{-j\angle H(e^{j\omega_0})} e^{-j(\omega_0 n + \phi)} \right]$$

$$= A |H(e^{j\omega_0})| \cos[\omega_0 n + \phi + \angle H(e^{j\omega_0})]$$

# LTI Systems in the Frequency Domain (cont.)

- **Example 3:** A sum of sinusoidal sequences

$$x[n] = \sum_{k=1}^K A_k \cos(\omega_k n + \phi_k)$$

$$y[n] = \sum_{k=1}^K A_k \underbrace{|H(e^{j\omega_k})|}_{\text{magnitude response}} \cos\left[\omega_k n + \phi_k + \underbrace{\angle H(e^{j\omega_k})}_{\text{phase response}}\right]$$

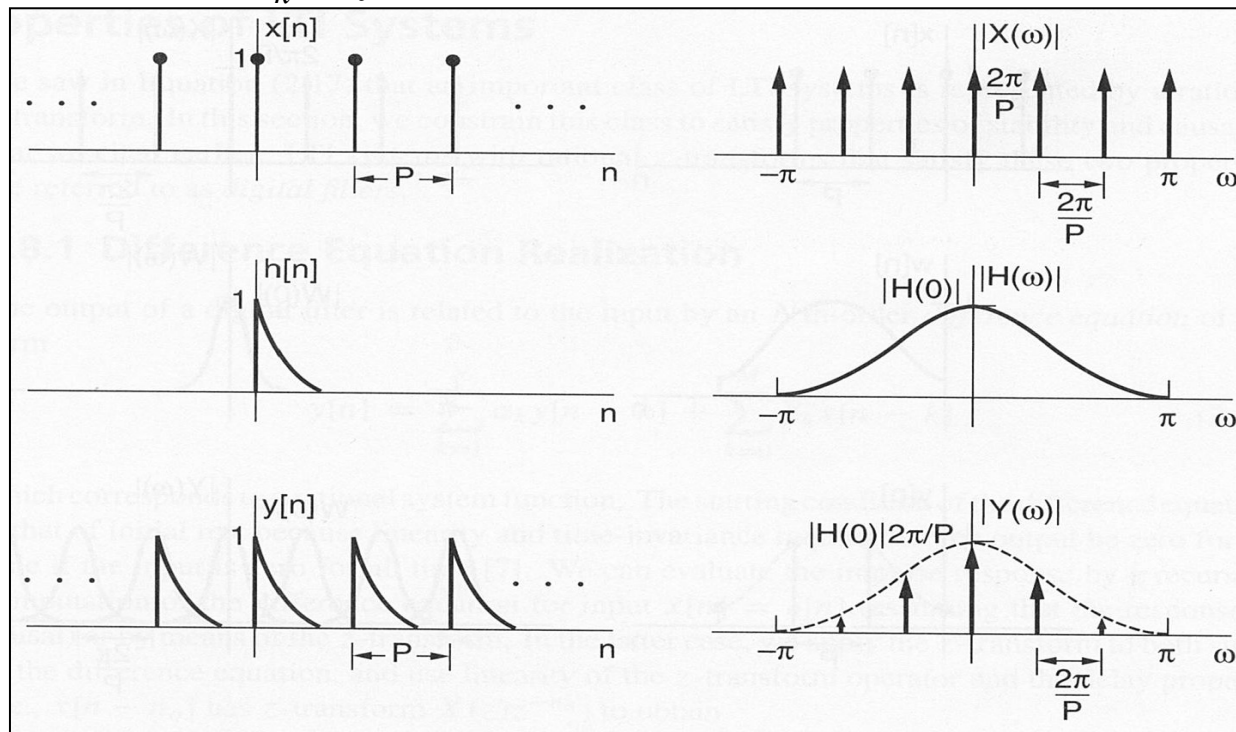
- A similar expression is obtained for an input consisting of a sum of complex exponentials

# LTI Systems in the Frequency Domain (cont.)

- Example 4: Convolution Theorem**  $x[n]*h[n] \Leftrightarrow X(e^{j\omega})H(e^{j\omega})$

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kP] \xrightarrow{\text{DTFT}} X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{P} \delta\left[\omega - \frac{2\pi}{P}k\right]$$

$$h[n] = \sum_{k=-\infty}^{\infty} a^n u[n], \quad |a| < 1 \xrightarrow{\text{DTFT}} H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$



$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$= \frac{1}{1 - ae^{-j\omega}} \sum_{k=-\infty}^{\infty} \frac{2\pi}{P} \delta\left[\omega - \frac{2\pi}{P}k\right]$$

$$= \frac{2\pi}{P} \frac{1}{1 - ae^{-j\omega}}$$

has a nonzero value when  $k = \frac{\omega P}{2\pi}$

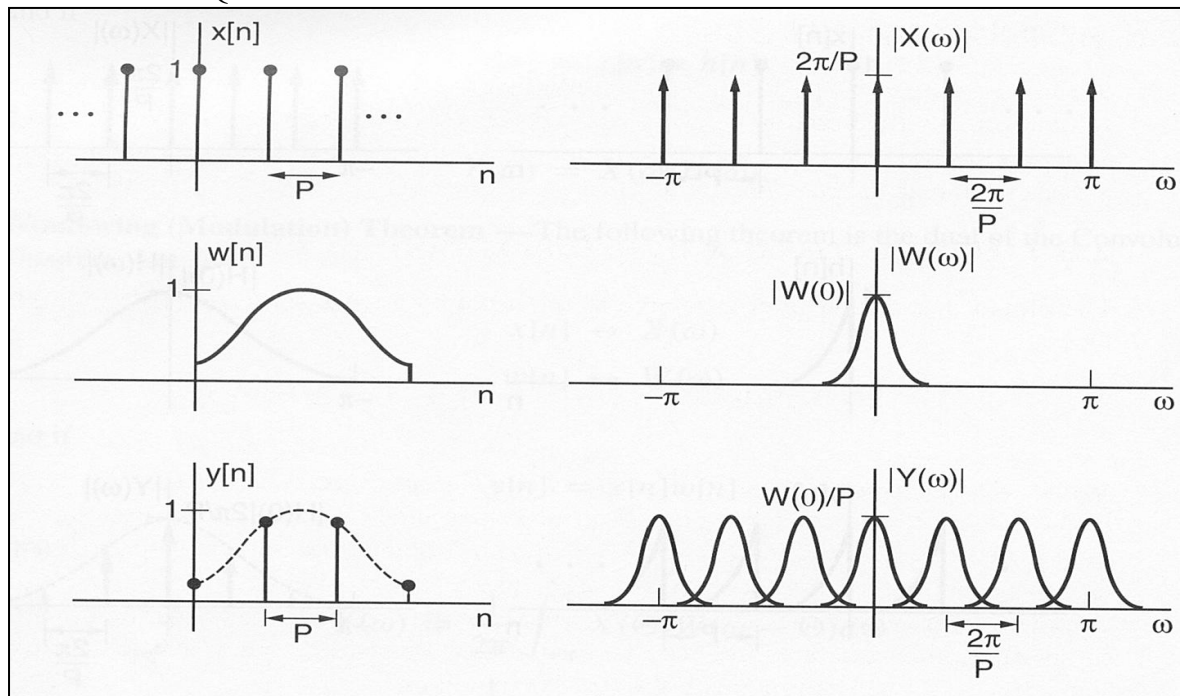
# LTI Systems in the Frequency Domain (cont.)

- Example 5: Windowing Theorem**  $x[n]w[n] \Leftrightarrow \frac{1}{2\pi} W(e^{j\omega}) * X(e^{j\omega})$

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kP]$$

$$w[n] = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right), & n = 0, 1, \dots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

Hamming window



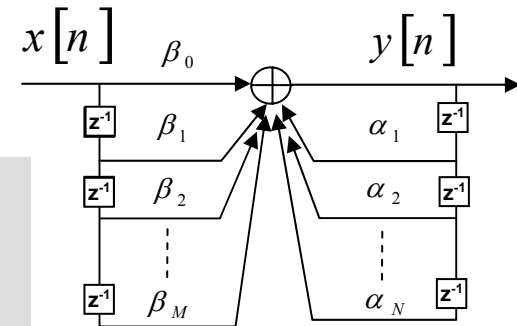
$$\begin{aligned} Y(e^{j\omega}) &= \frac{1}{2\pi} W(e^{j\omega}) * X(e^{j\omega}) \\ &= \frac{1}{2\pi} W(e^{j\omega}) * \sum_{k=-\infty}^{\infty} \frac{2\pi}{P} \delta\left(\omega - \frac{2\pi}{P}k\right) \\ &= \frac{1}{P} \sum_{k=-\infty}^{\infty} \left\{ W(e^{j\omega}) * \delta\left(\omega - \frac{2\pi}{P}k\right) \right\} \\ &= \frac{1}{P} \sum_{k=-\infty}^{\infty} \left\{ \sum_{m=-\infty}^{\infty} W(e^{jm}) \delta\left(\omega - \frac{2\pi}{P}k - m\right) \right\} \\ &= \frac{1}{P} \sum_{k=-\infty}^{\infty} W\left(e^{j\left(\omega - \frac{2\pi}{P}k\right)}\right) \end{aligned}$$

has a nonzero value when  $m = \omega - \frac{2\pi}{P}k$

# Difference Equation Realization for a Digital Filter

- The relation between the output and input of a digital filter can be expressed by

$$y[n] = \sum_{k=1}^N \alpha_k y[n-k] + \sum_{k=0}^M \beta_k x[n-k]$$



linearity and delay properties



**delay property**

$$x[n] \rightarrow X(z)$$

$$x[n - n_0] \rightarrow X(z)z^{-n_0}$$

$$Y(z) = \sum_{k=1}^N \alpha_k Y(z)z^{-k} + \sum_{k=0}^M \beta_k X(z)z^{-k}$$

A rational transfer function



$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M \beta_k z^{-k}}{1 - \sum_{k=1}^N \alpha_k z^{-k}}$$

**Causal:**

Rightsided, the ROC outside the outmost pole

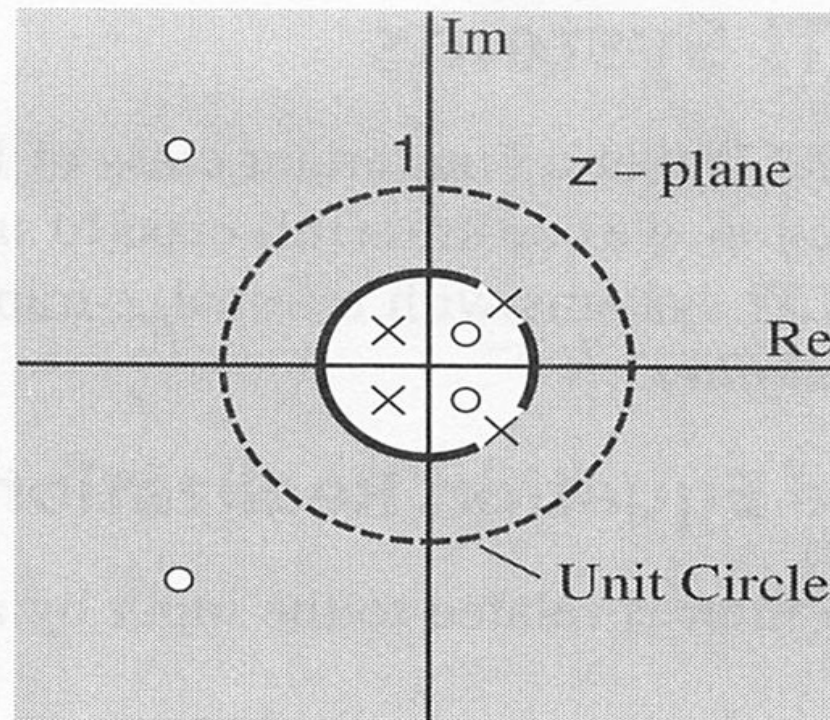
**Stable:**

The ROC includes the unit circle

**Causal and Stable:**

all poles must fall inside the unit circle (not including zeros)

# Difference Equation Realization for a Digital Filter (cont.)



**Figure 2.8** Pole-zero configuration for a causal and stable discrete-time system.

# Magnitude-Phase Relationship

- **Minimum phase system:**

- The z-transform of a system impulse response sequence ( a rational transfer function) has all zeros as well as poles inside the unit cycle
- Poles and zeros called “minimum phase components”
- **Maximum phase:** all zeros (or poles) outside the unit cycle

- **All-pass system:**

- Consist a cascade of factor of the form

$$\left[ \frac{1-a^* z}{1-az^{-1}} \right]^{\pm 1}$$

- Characterized by a frequency response with unit (or flat) magnitude for all frequencies  $\left| \frac{1-a^* z}{1-az^{-1}} \right| = 1$

Poles and zeros occur at conjugate reciprocal locations

## Magnitude-Phase Relationship (cont.)

- Any digital filter can be represented by the cascade of a minimum-phase system and an all-pass system

$$H(z) = H_{\min}(z)H_{ap}(z)$$

Suppose that  $H(z)$  has only one zero  $1/a^*$  ( $|a| < 1$ )

outside the unit circle.  $H(z)$  can be expressed as :

$$H(z) = H_1(z)(1 - a^*z) \quad (H_1(z) \text{ is a minimum phase filter})$$

$$= H_1(z)(1 - az^{-1}) \frac{(1 - a^*z)}{(1 - az^{-1})}$$

where :

$H_1(z)(1 - az^{-1})$  is also a minimum phase filter.

$\frac{(1 - a^*z)}{(1 - az^{-1})}$  is a all - pass filter.



# FIR Filters

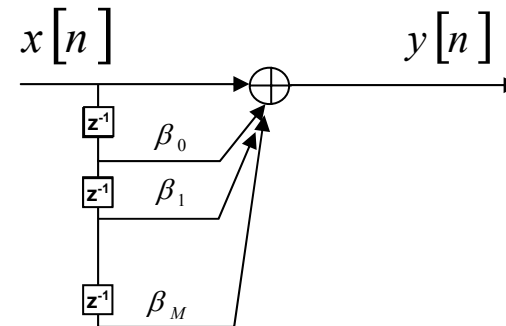
- FIR (Finite Impulse Response)

- The impulse response of an FIR filter has finite duration
- Have no denominator in the rational function
  - No feedback in the difference equation

$$H(z)$$

$$y[n] = \sum_{r=0}^M \beta_r x[n-r]$$
$$h[n] = \begin{cases} \beta_n, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$$

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^M \beta_k z^{-k}$$



- Can be implemented with simple a train of delay, multiple, and add operations

# First-Order FIR Filters

- A special case of FIR filters

$$y[n] = x[n] + \alpha x[n-1] \iff H(z) = 1 + \alpha z^{-1}$$

$$|H(e^{j\omega})|^2 = |1 + \alpha(\cos \omega - j \sin \omega)|^2 \quad H(e^{j\omega}) = 1 + \alpha e^{-j\omega}$$

$$= (1 + \alpha \cos \omega)^2 + (\alpha \sin \omega)^2 = 1 + 2\alpha \cos \omega$$

$$\theta(e^{j\omega}) = -\arctan\left(\frac{\alpha \sin \omega}{1 + \alpha \cos \omega}\right) \quad \alpha < 0 : \text{pre-emphasis filter}$$

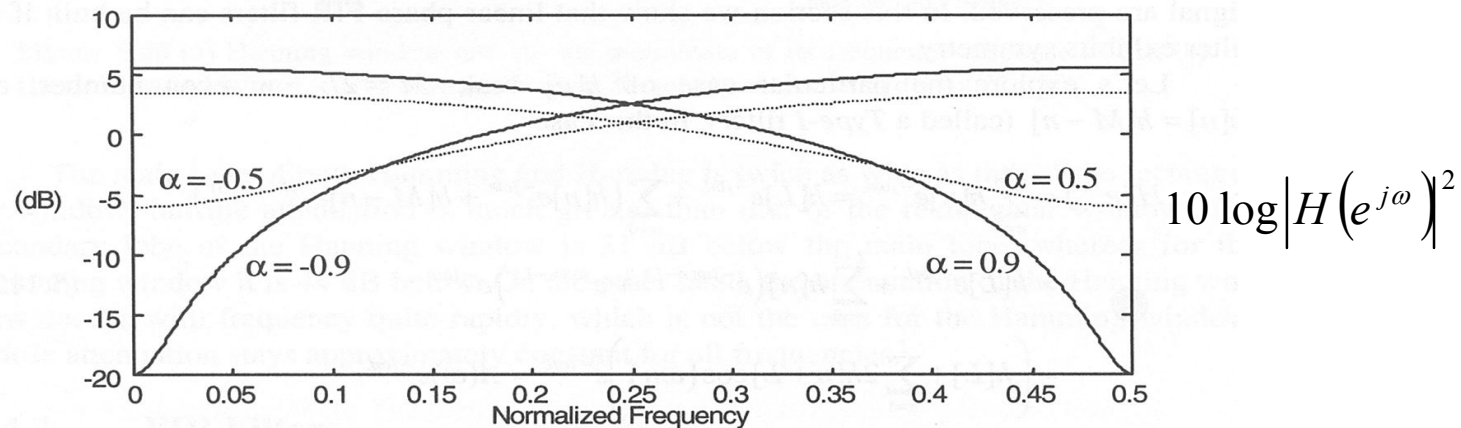


Figure 5.21 Frequency response of the first order FIR filter for various values of  $\alpha$ .

# Discrete Fourier Transform (DFT)

- The Fourier transform of a discrete-time sequence is a continuous function of frequency
  - We need to sample the Fourier transform finely enough to be able to recover the sequence
  - For a sequence of finite length  $N$ , sampling yields the new transform referred to as *discrete Fourier transform* (DFT)

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}, \quad 0 \leq n \leq N-1$$

DFT, **Analysis**

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn}, \quad 0 \leq n \leq N-1$$

Inverse DFT, **Synthesis**

# Discrete Fourier Transform (cont.)

$$\forall 0 \leq k \leq N-1$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}, \quad 0 \leq n \leq N-1$$

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & e^{-j\frac{2\pi}{N}1 \cdot 1} & \dots & e^{-j\frac{2\pi}{N}1 \cdot (N-1)} \\ \vdots & \vdots & \dots & \vdots \\ 1 & e^{-j\frac{2\pi}{N}(N-1) \cdot 1} & \dots & e^{-j\frac{2\pi}{N}(N-1) \cdot (N-1)} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} = \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$$

# Discrete Fourier Transform (cont.)

- Orthogonality of Complex Exponentials

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-r)n} = \begin{cases} 1, & \text{if } k-r = mN \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} x[n] &= \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn} \\ \Rightarrow \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}rn} &= \sum_{n=0}^{N-1} \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}(k-r)n} \\ &= \sum_{k=0}^{N-1} X[k] \left[ \frac{1}{N} \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-r)n} \right] \\ &= X[r] \\ \Rightarrow X[k] &= \sum_{r=0}^{N-1} x[r] e^{-j\frac{2\pi}{N}kr} \end{aligned}$$

$$\begin{aligned} X[k] &= X[r + mN] \\ &= X[r] \end{aligned}$$

# Discrete Fourier Transform (DFT)

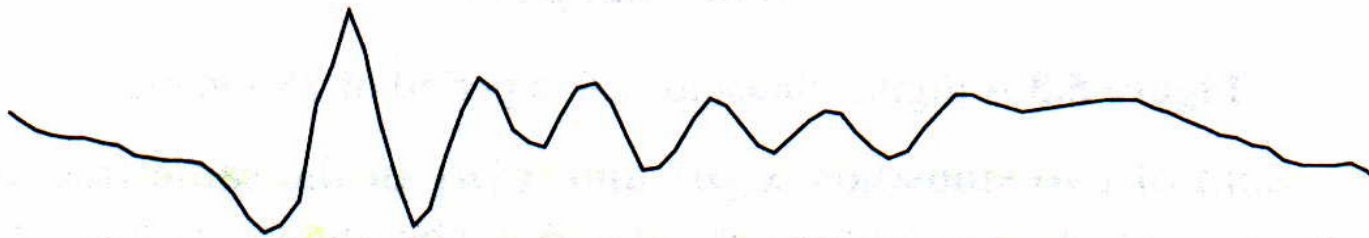
- Parseval's theorem

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

Energy density

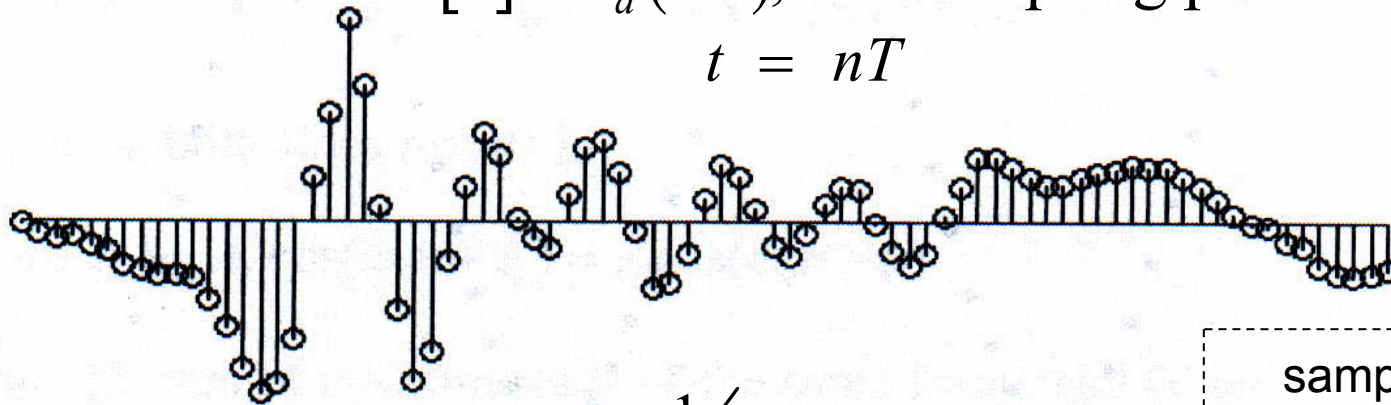
# Analog Signal to Digital Signal

Analog Signal



Discrete-time Signal or Digital Signal

$$x[n] = x_a(nT), \quad T : \text{sampling period} \\ t = nT$$

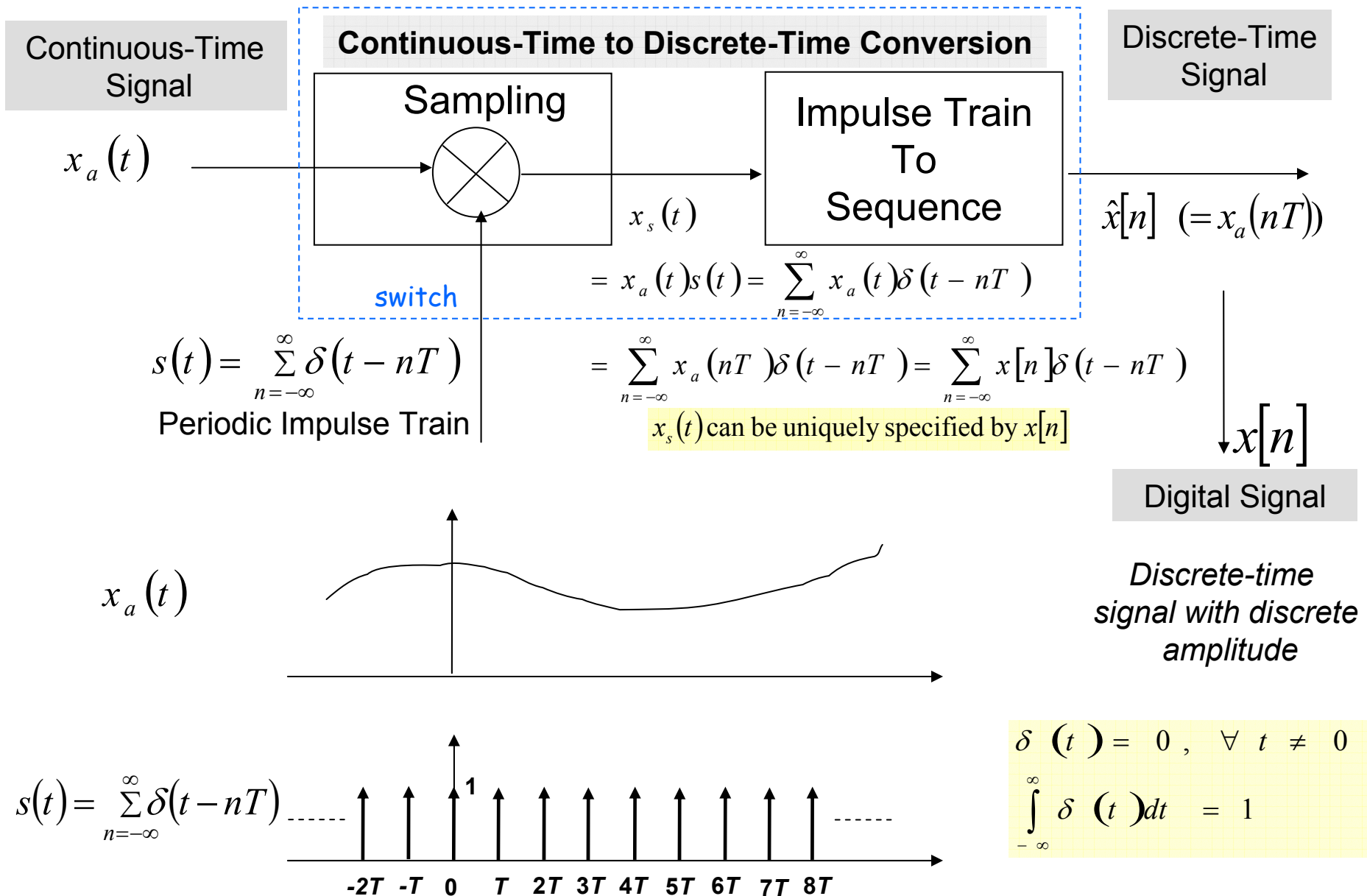


Digital Signal:  
*Discrete-time  
signal with discrete  
amplitude*

$$F_s = \frac{1}{T} \text{ sampling rate}$$

sampling period =  $125 \mu\text{s}$   
=> sampling rate = 8kHz

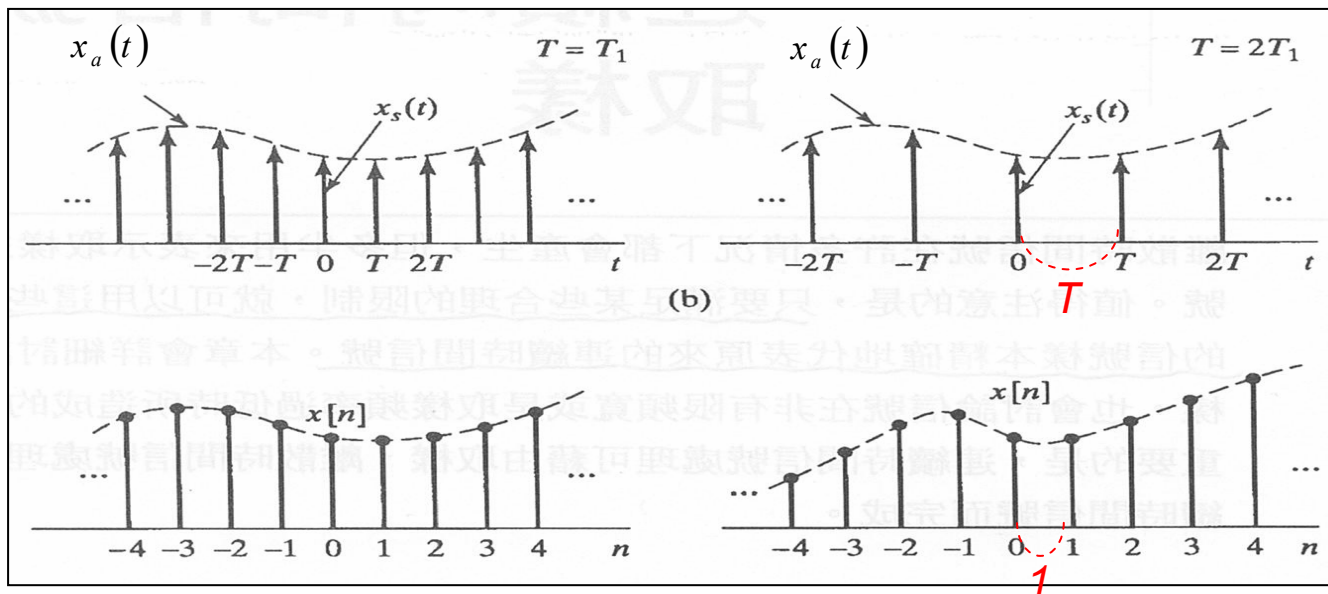
# Analog Signal to Digital Signal (cont.)





# Analog Signal to Digital Signal (cont.)

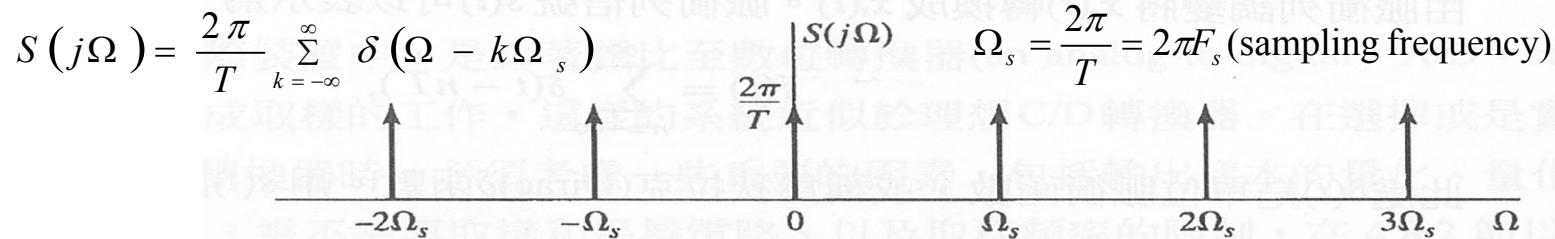
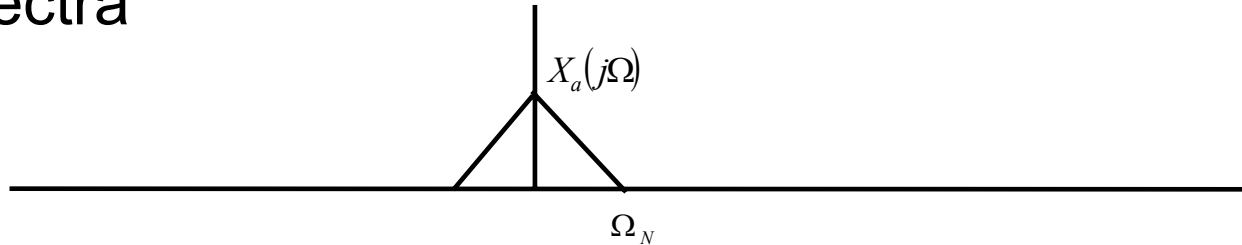
- A continuous signal sampled at different periods



$$\begin{aligned}
 x_s(t) &= x_a(t)s(t) = \sum_{n=-\infty}^{\infty} x_a(t)\delta(t - nT) \\
 &= \sum_{n=-\infty}^{\infty} x_a(nT)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)
 \end{aligned}$$

# Analog Signal to Digital Signal (cont.)

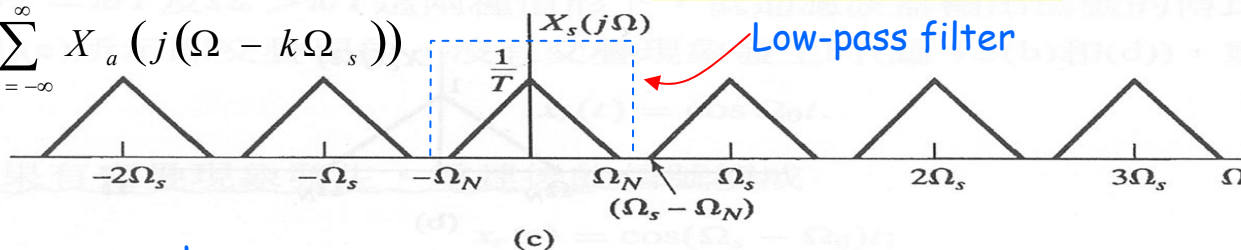
- Spectra



$$X_s(j\Omega) = \frac{1}{2\pi} X_a(j\Omega) * S(j\Omega)$$

$$R_{\Omega_s}(j\Omega) = \begin{cases} T & |\Omega| < \Omega_s / 2 \\ 0 & \text{otherwise} \end{cases} \Rightarrow X_a(j\Omega) = R_{\Omega_s}(j\Omega) X_p(j\Omega)$$

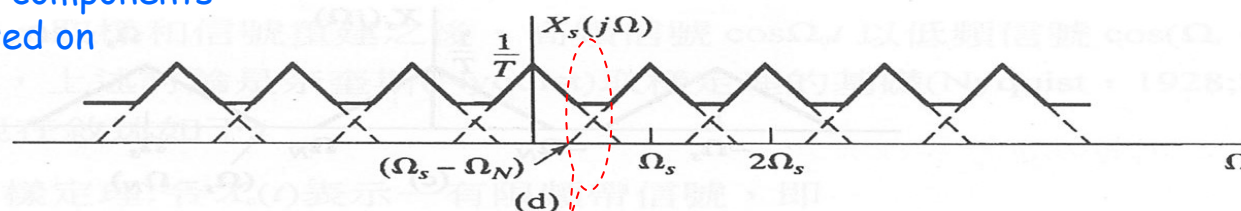
$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(j(\Omega - k\Omega_s))$$



$$\Omega_N < \frac{1}{2} \Omega_s \left( = \frac{\pi}{T} \right)$$

$$\left\{ \begin{array}{l} \because (\Omega_s - \Omega_N) > \Omega_N \\ \Rightarrow \Omega_s > 2\Omega_N \end{array} \right\}$$

high frequency components  
got superimposed on  
low frequency  
components



$$\Omega_N > \frac{1}{2} \Omega_s \left( = \frac{\pi}{T} \right)$$

$$\left\{ \begin{array}{l} \because (\Omega_s - \Omega_N) < \Omega_N \\ \Rightarrow \Omega_s < 2\Omega_N \end{array} \right\}$$

**aliasing distortion**

$\Rightarrow X_a(j\Omega)$  can't be recovered from  $X_p(j\Omega)$

# Analog Signal to Digital Signal (cont.)

- To avoid aliasing (*overlapping, fold over*)
  - The sampling frequency should be greater than two times of frequency of the signal to be sampled  $\rightarrow \Omega_s > 2\Omega_N$
  - (Nyquist) sampling theorem
- To reconstruct the original continuous signal
  - Filtered with a low pass filter with band limit  $\Omega_s$ 
    - Convolved in time domain

