Continuous Random Variables: Derived Distributions

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Reference:

- D. P. Bertsekas, J. N. Tsitsiklis, *Introduction to Probability* , Section 3.6

Two-step approach to Calculating Derived PDF

- Calculate the PDF of a Function $Y = g(X)$ of a continuous random variable X

1. Calculate the CDF F_Y of Y using the formula

$$F_Y(y) = \mathbf{P}(g(X) \le y) = \int_{\{x \mid g(x) \le y\}} f_X(x)\,dx$$

2. Differentiate to obtain the PDF (called the derived distribution) of Y

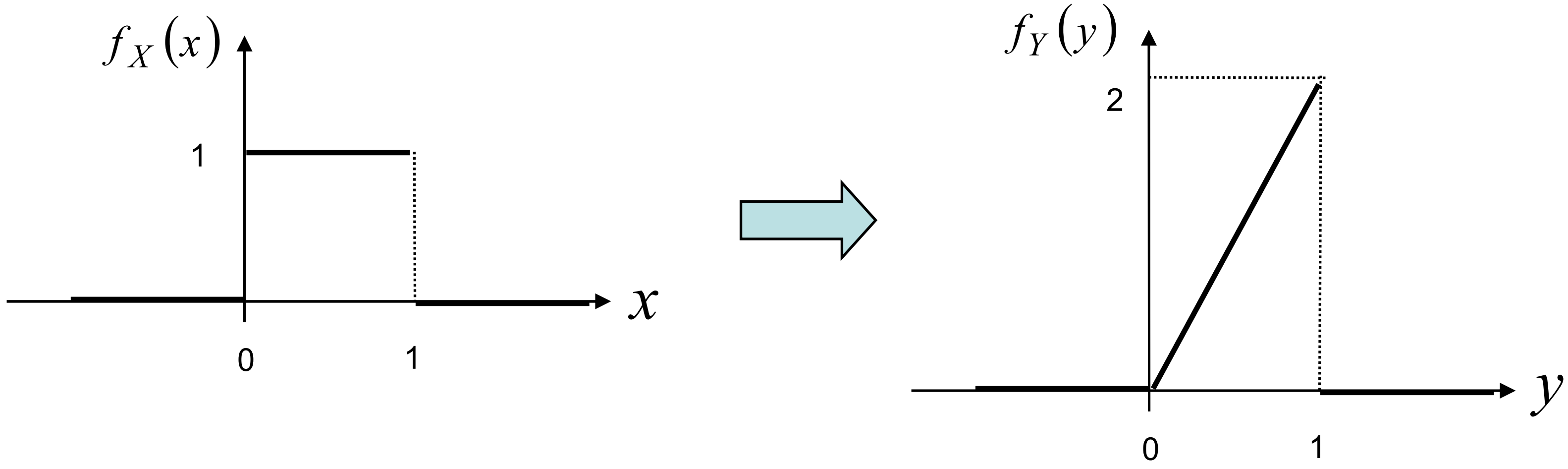
$$f_Y(y) = \frac{dF_Y(y)}{dy}$$

Illustrative Examples (1/2)

- **Example 3.20.** Let X be uniform on [0, 1]. Find the PDF of $Y = \sqrt{X}$. Note that Y takes values between 0 and 1.

$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}\left(\sqrt{X} \le y\right) = \mathbf{P}\left(X \le y^2\right) = y^2$$

$$\therefore\ f_Y(y) = \frac{dF_Y(y)}{dy} = 2y, \qquad 0 \le y \le 1$$



Illustrative Examples (2/2)

- **Example 3.22.** Let $Y = X^2$, where X is a random variable with known PDF $f_X(x)$. Find the PDF of Y represented in terms of $f_X(x)$.

For any $y \geq 0$, we have

$$
\begin{aligned}
F_Y(y) &= \mathbf{P}(Y \leq y) = \mathbf{P}\left(X^2 \leq y\right) \\
&= \mathbf{P}\left(-\sqrt{y} \leq X \leq \sqrt{y}\right) \\
&= F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad f_Y(y) &= \frac{dF_Y(y)}{dy} \\
&= \left[\frac{dF_X\left(\sqrt{y}\right)}{d\sqrt{y}} \cdot \frac{d\sqrt{y}}{dy}\right] - \left[\frac{dF_X\left(-\sqrt{y}\right)}{d\left(-\sqrt{y}\right)} \cdot \frac{d\left(-\sqrt{y}\right)}{dy}\right] \\
&= \frac{1}{2\sqrt{y}} f_X\left(\sqrt{y}\right) + \frac{1}{2\sqrt{y}} f_X\left(-\sqrt{y}\right) \\
&= \frac{1}{2\sqrt{y}}\left[f_X\left(\sqrt{y}\right) + f_X\left(-\sqrt{y}\right)\right]
\end{aligned}
$$

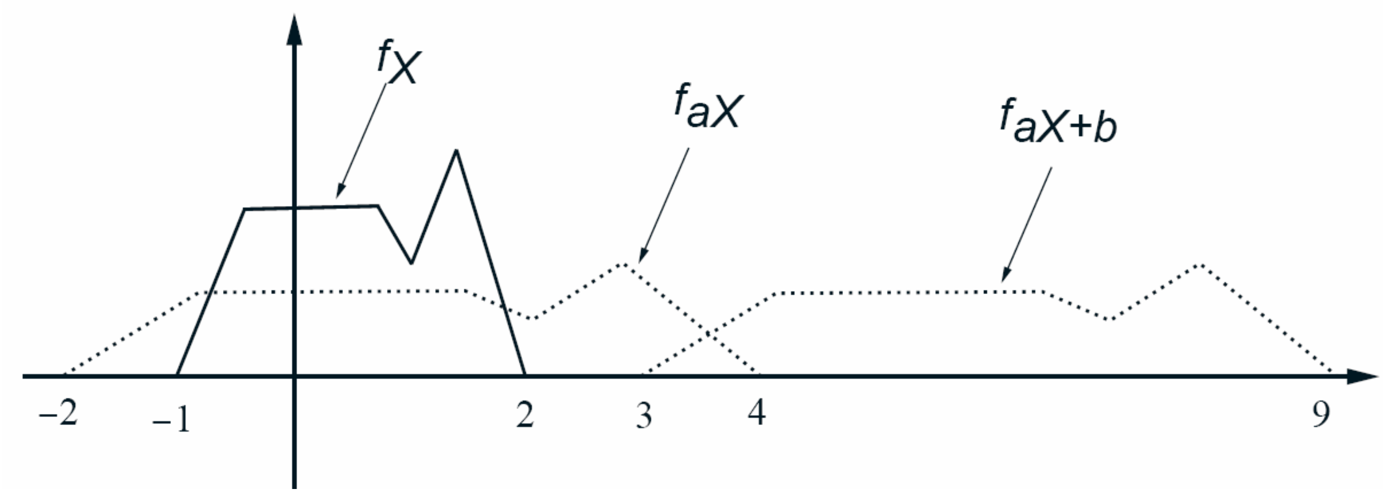
The PDF of a Linear Function of a Random Variable

- Let X be a continuous random variable with PDF $f_X(x)$, and let

$$Y = aX + b,$$

for some scalar $a \neq 0$ and b. Then,

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$



a>0, b>0

The PDF of a Linear Function of a Random Variable (1/2)

- Verification of the above formula

$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}(aX + b \le y)$$

(i) For $a > 0$

$$F_Y(y) = \mathbf{P}\left(X \le \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right)$$

$$\Rightarrow f_Y(y) = \frac{dF_X\left(\frac{y-b}{a}\right)}{d\left(\frac{y-b}{a}\right)} \cdot \frac{d\left(\frac{y-b}{a}\right)}{dy} = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

(ii) For $a < 0$

$$F_Y(y) = \mathbf{P}\left(X \ge \frac{y-b}{a}\right) = 1 - F_X\left(\frac{y-b}{a}\right)$$

$$\Rightarrow f_Y(y) = -\frac{dF_X\left(\frac{y-b}{a}\right)}{d\left(\frac{y-b}{a}\right)} \cdot \frac{d\left(\frac{y-b}{a}\right)}{dy} = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

$$\therefore f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

Illustrative Examples (1/2)

- **Example 3.23. A linear function of an exponential random variable.**
  - Suppose that X is an exponential random variable with PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

    - where λ is a positive parameter. Let $Y = aX + b$. Then,

$$f_Y(y) = \begin{cases} \frac{1}{|a|} \lambda e^{-\lambda (y-b)/a}, & \text{if } (y-b)/a \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Illustrative Examples (2/2)

- **Example 3.24. A linear function of a normal random variable is normal.**
  - Suppose that X is a normal random variable with mean μ and variance σ^2,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty \le x \le \infty$$

  - And let $Y = aX + b$, where a and b are some scalars. We have

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$$= \frac{1}{|a|} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{\left(\frac{y-b}{a}-\mu\right)^2}{2\sigma^2}}$$

$$== \frac{1}{\sqrt{2\pi}\,|a|\sigma} e^{-\frac{(y-(b+a\mu))^2}{2a^2\sigma^2}}, \quad -\infty \le y \le \infty$$

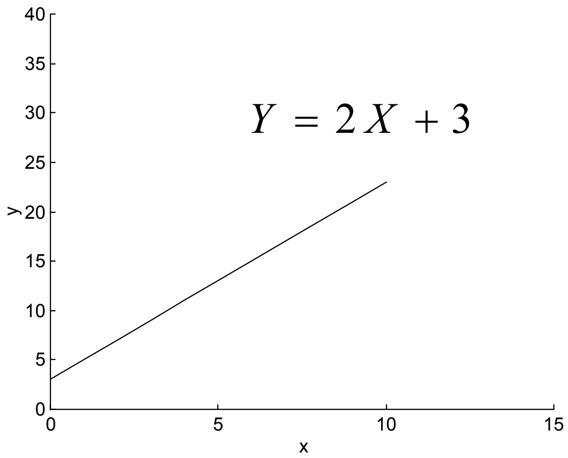
$\therefore$ Y is also a normal random variable with mean $a\mu + b$ and variance $a^2\sigma^2$

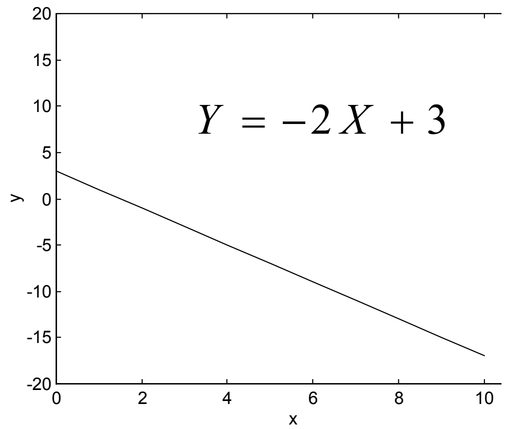
Monotonic Functions of a Random Variable (1/4)

- Let X be a continuous random variable and have values in a certain interval I ($f_X(x)=0 \quad \text{for } x \notin I$). While random variable $Y=g(X)$ and we assume that g is **strictly monotonic** over the interval I . That is, either

  (1) $g(x)<g(x')$ for all $x,x' \in I$, satisfying $x<x'$ (monotonically increasing case), or

  (2) $g(x)>g(x')$ for all $x,x' \in I$, satisfying $x<x'$ (monotonically decreasing case)





Monotonic Functions of a Random Variable (2/4)

- Suppose that g is monotonic and that for some function h and all x in the range I of X we have

$$y = g(x) \quad \text{if and only if } x = h(y)$$

  – For example,

$$y = g(x) = ax + b \qquad \Rightarrow \qquad x = h(y) = \frac{y-b}{a}$$

$$y = g(x) = e^{ax} \qquad \Rightarrow \qquad x = h(y) = \frac{\ln y}{a}$$

$$y = g(x) = -ax + b \qquad \Rightarrow \qquad x = h(y) = -\frac{y-b}{a}$$

Monotonic Functions of a Random Variable (3/4)

– Assume that h has first derivative $\frac{dh(y)}{dy}$. Then the PDF of Y in the region where $f_Y(y) > 0$ is given by

$$f_Y(y) = f_X(h(y))\left|\frac{dh(y)}{dy}\right|$$

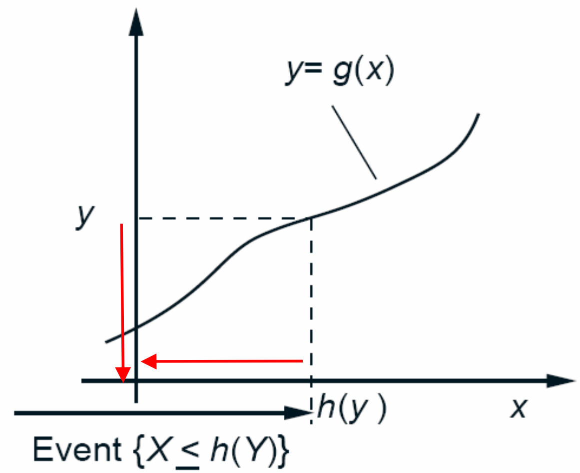
- For the monotonically increasing case

$$F_Y(y) = \mathbf{P}(g(X) \le y) = \mathbf{P}(X \le h(y)) = F_X(h(y))$$

$$\Rightarrow f_Y(y) = \frac{dF_Y(y)}{dy}$$

$$= \frac{dF_X(h(y))}{dy} = \frac{dF_X(h(y))}{dh(y)} \cdot \frac{dh(y)}{dy}$$

$$= f_X(h(y)) \cdot \frac{dh(y)}{dy}$$



Monotonic Functions of a Random Variable (4/4)

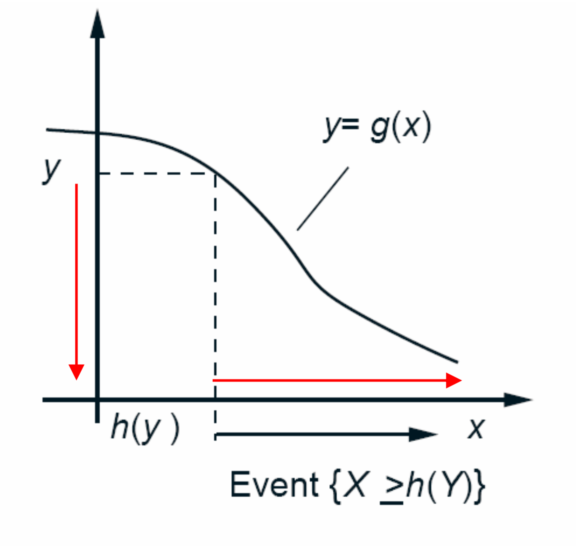
- For the monotonically decreasing case

$$F_Y(y) = \mathbf{P}(g(X) \le y) = \mathbf{P}(X \ge h(y)) = 1 - F_X(h(y))$$

$$\Rightarrow f_Y(y) = \frac{dF_Y(y)}{dy}$$

$$= -\frac{dF_X(h(y))}{dy} = -\frac{dF_X(h(y))}{dh(y)} \cdot \frac{dh(y)}{dy}$$

$$= -f_X(h(y)) \cdot \frac{dh(y)}{dy}$$



Illustrative Examples (1/3)

- **Example 3.25.** Let $Y = g(X) = X^2$, where X is a continuous uniform random variable in the interval $(0, 1]$.
  - What is the PDF of y ?
    - Within this interval, g is strictly monotonic, and its inverse is $h(y) = \sqrt{y}$

We have

$$f_X(x) = 1 \quad \text{for all} \quad 0 < x \leq 1$$

and $g(X)$ being strictly increasing

$\Rightarrow$

$$f_X(\sqrt{y}) = 1, \qquad \text{for all } 0 < y \leq 1$$

$$\therefore \quad f_Y(y) = \frac{dh(y)}{dy} f_X(\sqrt{y}) = \begin{cases} \dfrac{1}{2\sqrt{y}}, & \text{if } y \in (0, 1] \\ 0, & \text{otherwise} \end{cases}$$

Illustrative Examples (2/3)

- **Example 3.26.** Let X and Y be independent random variables that are uniformly distributed on the interval [0, 1], respectively. What is the PDF of the random variable $Z = \max\{X, Y\}$

$$
\begin{aligned}
F_Z(z) &= \mathbf{P}(\max\{X, Y\} \le z) \\
&= \mathbf{P}(X \le z, Y \le z) \\
&= \mathbf{P}(Y \le z)\mathbf{P}(Y \le z) \\
&= z^2
\end{aligned}
$$

$$
\therefore\ f_Z(z) = \begin{cases} 2z, & \text{if } 0 \le z \le 1 \\ 0, & \text{otherwise} \end{cases}
$$

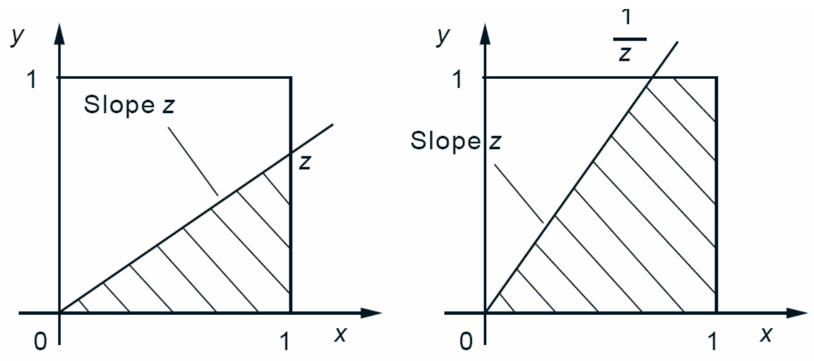
Illustrative Examples (3/3)

- **Example 3.27.** Let X and Y be independent random variables that are uniformly distributed on the interval [0, 1]. What is the PDF of the random variable $Z = Y / X$

$\because X, Y$ are independent

$\therefore f_{X,Y}(x, y) = f_X(x) f_Y(y) = 1,$

for all $x, y,\ 0 \le x, y \le 1$



$$F_Z(z) = \mathbf{P}(Y / X \le z)$$

$$= \begin{cases} z/2, & \text{if } 0 \le z \le 1 \\ 1 - (1/2z), & \text{if } z > 1 \\ 0, & \text{otherwise} \end{cases} \Rightarrow f_Z(z) = \begin{cases} 1/2, & \text{if } 0 \le z \le 1 \\ 1/(2z^2), & \text{if } z > 1 \\ 0, & \text{otherwise} \end{cases}$$

Recitation

- SECTION 3.6 Derived Functions
  - Problems 30, 31, 36, 38, 39