Chapter 4
General Vector Spaces

Outline

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4.1

Real Vector Spaces

Definition (Vector Space)

- Let *V* be an arbitrary nonempty set of objects on which two operations are defined:
  - Addition
  - Multiplication by scalars
- If the following *axioms* (公理) are satisfied by all objects **u**, **v**, **w** in *V* and all scalars *k* and *l*, then we call *V* a vector space (向量空間) and we call the objects in *V* vectors.

Definition (Vector Space)

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u} + \mathbf{v}$ is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. There is an object $\mathbf{0}$ in V, called a zero vector for V, such that $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u}$ in V.
5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a negative of $\mathbf{u}$, such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$.
6. If k is any scalar and $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V.
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$
9. $k(l\mathbf{u}) = (kl)(\mathbf{u})$
10. $1\mathbf{u} = \mathbf{u}$

Remarks

- Depending on the application, *scalars* may be real numbers or complex numbers.
  - Vector spaces in which the scalars are complex numbers are called complex vector spaces (複數向量空間), and those in which the scalars must be real are called real vector spaces (實數向量空間).
- The definition of a vector space specifies neither the nature of the vectors nor the operations.
  - *Any kind of object can be a vector*, and the operations of addition and scalar multiplication may not have any relationship or similarity to the standard vector operations on R^n.
  - *The only requirement is that the ten vector space axioms be satisfied.*

Show a Set as a Vector Space

- Step 1: Identify the set V of objects that will become vectors.
- Step 2: Identify the addition and scalar multiplication operations on V.
- Step 3. Verify Axioms 1 and 6. Axiom 1 is called ***closure under addition*** (加法封閉性), and Axiom 6 is call ***closure under scalar multiplication*** (純量乘法封閉性).
- Step 4: Confirm that Axioms 2, 3, 4, 5, 7, 8, 9, and 10 hold.

Example: The Zero Vector Space

- Let V consist of a single object, which we denote by $\mathbf{0}$, and define $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $k\mathbf{0} = \mathbf{0}$ for all scalars k.
- It's easy to check that all the vector space axioms are satisfied.
- We call this the ***zero vector space***.

Example (R^n Is a Vector Space)

- The set $V = R^n$ with the standard operations of addition and scalar multiplication is a vector space.

- Axioms 1 and 6 follow from the definitions of the standard operations on R^n; the remaining axioms follow from Theorem 3.1.1.

- The three most important special cases of R^n are R (the real numbers), R^2 (the vectors in the plane), and R^3 (the vectors in 3-space).

Example (2×2 Matrices)

- Show that the set V of all 2×2 matrices with real entries is a vector space <u>if vector addition is defined to be matrix addition and vector scalar multiplication is defined to be matrix scalar multiplication</u>.

- Let $\mathbf{u} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$

- To prove Axiom 1, we must show that $\mathbf{u} + \mathbf{v}$ is an object in V; that is, we must show that $\mathbf{u} + \mathbf{v}$ is a 2×2 matrix.

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix}$$

Example

- Similarly, Axiom 6 hold because for any real number k we have

$$k\mathbf{u} = k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$

  so that $k\mathbf{u}$ is a 2×2 matrix and consequently is an object in V.

- Axioms 2 follows from Theorem 1.4.1a since

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{v}+\mathbf{u}$$

- Similarly, Axiom 3 follows from part (b) of that theorem; and Axioms 7, 8, and 9 follow from part (h), (j), and (e), respectively.

Example

- To prove Axiom 4, let $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

  Then
  $$\mathbf{0} + \mathbf{u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}$$
  Similarly, $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
- To prove Axiom 5, let $-\mathbf{u} = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$

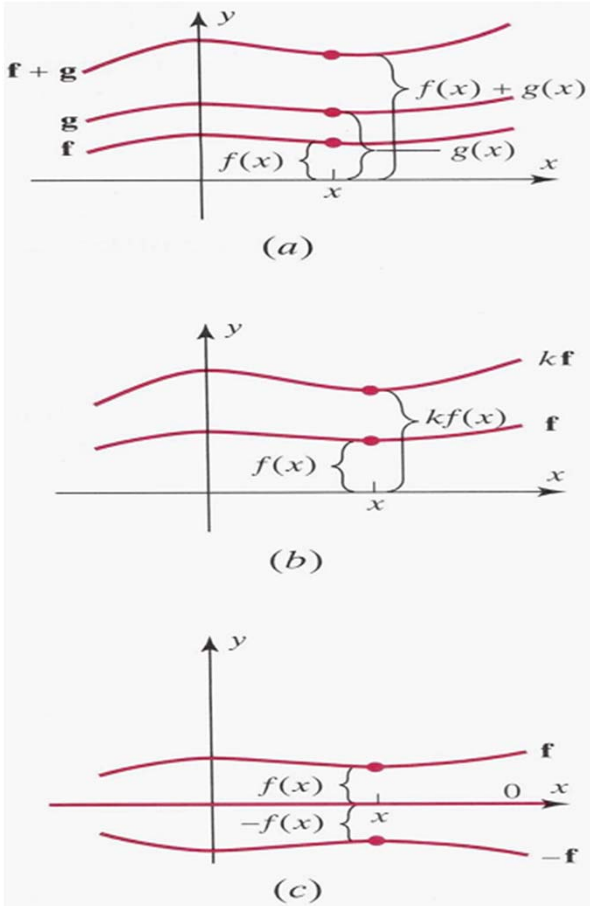
  Then
  $$\mathbf{u} + (-\mathbf{u}) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$
  Similarly, $(-\mathbf{u}) + \mathbf{u} = \mathbf{0}$.
- For Axiom 10, $1\mathbf{u} = \mathbf{u}$.

Example (Vector Space of $m \times n$ Matrices)

- The previous example is a special case of a more general class of vector spaces.
- The arguments in that example can be adapted to show that the set V of all $m \times n$ matrices with real entries, together with the operations matrix addition and scalar multiplication, is a vector space.
- The $m \times n$ zero matrix is the zero vector **0**, and if **u** is the $m \times n$ matrix U, then matrix $-U$ is the negative $-$**u** of the vector **u**.
- We shall denote this vector space by the symbol M_{mn}

Example (Vector Space of Real-Valued Functions)

- Let V be the set of real-valued functions defined on the entire real line $(-\infty,\infty)$. If $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ are two such functions and k is any real number, defined the sum function $\mathbf{f} + \mathbf{g}$ and the scalar multiple $k\,\mathbf{f}$, respectively, by $(\mathbf{f} + \mathbf{g})(x) = f(x) + g(x)$ and $(k\,\mathbf{f})(x)=kf(x)$.
- In other words, the value of the function $\mathbf{f} + \mathbf{g}$ at x is obtained by adding together the values of $\mathbf{f}$ and $\mathbf{g}$ at x (Figure 4.1.1 a). Similarly, the value of $k\,\mathbf{f}$ at x is k times the value of $\mathbf{f}$ at x (Figure 4.1.1 b). This vector space is denoted by $F(-\infty,\infty)$. If $\mathbf{f}$ and $\mathbf{g}$ are vectors in this space, then to say that $\mathbf{f} = \mathbf{g}$ is equivalent to saying that $f(x) = g(x)$ for all x in the interval $(-\infty,\infty)$.
- The vector $\mathbf{0}$ in $F(-\infty,\infty)$ is the constant function that identically zero for all value of x. The negative of a vector $\mathbf{f}$ is the function $-\mathbf{f} = -f(x)$. Geometrically, the graph of $-\mathbf{f}$ is the reflection of the graph of $\mathbf{f}$ across the x-axis (Figure 4.1.1.c).



Example (Not a Vector Space)

- Let $V = R^2$ and define addition and scalar multiplication operations as follows: If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, then define

  $$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$

  and if k is any real number, then define

  $$k\,\mathbf{u} = (k\,u_1, 0)$$

- There are values of $\mathbf{u}$ for which Axiom 10 fails to hold. For example, if $\mathbf{u} = (u_1, u_2)$ is such that $u_2 \neq 0$,then

  $$1\mathbf{u} = 1\,(u_1, u_2) = (1\;u_1, 0) = (u_1, 0) \neq \mathbf{u}$$

- Thus, V is *not* a vector space with the stated operations.

An Unusual Vector Space

- Let V be the set of positive real numbers, and define the operation on V to be

$$u+v = uv,\ ku = u^k$$

- For example: $1+1 = 1$ and $2(1) = 1^2 = 1$
- The set V with these operations satisfies the 10 vector space axioms and hence is a vector space!
  - Axiom 4: the zero vector in this space is the number 1 since $u+1=u$
  - Axiom 5: the negative of a vector u is its reciprocal ($-u = 1/u$) since $u+(1/u)=u(1/u) = 1 = \mathbf{0}$
  - Axiom 7: $k(u+v) = (uv)^k = u^k v^k = (ku) + (kv)$

Every Plane Through the Origin Is a Vector Space

- *Check all the axioms!*
  - Let V be any plane through the origin in R^3. Since R^3 itself is a vector space, Axioms 2, 3, 7, 8, 9, and 10 hold for all points in R^3 and consequently for all points in the plane V.
  - We need only show that Axioms 1, 4, 5, and 6 are satisfied.

Every Plane Through the Origin Is a Vector Space

- Since the plane V passes through the origin, it has an equation of the form $ax + by + cz = 0$. If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are points in V, then $au_1 + bu_2 + cu_3 = 0$ and $av_1 + bv_2 + cv_3 = 0$. Adding these equations gives $a(u_1 + v_1) + b(u_2 + v_2) + c(u_3 + v_3) = 0$.
- Axiom 1: $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$; thus $\mathbf{u} + \mathbf{v}$ lies in the plane V.
- Axioms 5: Multiplying $au_1 + bu_2 + cu_3 = 0$ through by -1 gives $a(-u_1) + b(-u_2) + c(-u_3) = 0$; thus, $-\mathbf{u} = (-u_1, -u_2, -u_3)$ lies in V.

Theorem 4.1.1

- Let V be a vector space, $\mathbf{u}$ be a vector in V, and k a scalar; then:
  - $0\,\mathbf{u} = \mathbf{0}$
  - $k\,\mathbf{0} = \mathbf{0}$
  - $(-1)\,\mathbf{u} = -\mathbf{u}$
  - If $k\,\mathbf{u} = \mathbf{0}$, then $k = 0$ or $\mathbf{u} = \mathbf{0}$.

Proof of Theorem 4.1.1(a)

- We can write

$$0\mathbf{u} + 0\mathbf{u} = (0+0)\mathbf{u} \text{ [Axiom 8]}$$

$$= 0\mathbf{u} \text{ [Property of the number 0]}$$

- By Axiom 5 the vector 0**u** has a negative, -0**u**. Adding this negative to both sides above yields

$$[0\mathbf{u} + 0\mathbf{u}] + (-0\mathbf{u}) = (0+0)\mathbf{u} + (-0\mathbf{u}) \text{ [Axiom 3]}$$

$$0\mathbf{u} + \mathbf{0} = \mathbf{0} \text{ [Axiom 5]}$$

$$0\mathbf{u} = \mathbf{0} \text{ [Axiom 4]}$$

Proof of Theorem 4.1.1(c)

- To show that $(-1)\mathbf{u} = -\mathbf{u}$, we must demonstrate that $\mathbf{u} + (-1)\mathbf{u} = \mathbf{0}$.
- To see this, we observe that

$$\mathbf{u} + (-1)\mathbf{u} = 1\mathbf{u} + (-1)\mathbf{u} \text{ [Axiom 10]}$$

$$= (1 + (-1))\mathbf{u} \text{ [Axiom 8]}$$

$$= 0\mathbf{u} \text{ [Property of numbers]}$$

$$= \mathbf{0} \text{ [Property (a) above]}$$

4.2

Subspaces

Subspaces (子空間)

- Definition
  - A subset W of a vector space V is called a subspace of V if W is itself a vector space under the addition and scalar multiplication defined on V.
- In general, one must verify the ten vector space axioms to show that a set W with addition and scalar multiplication forms a vector space.
- However, some axioms are inherited from V. For example, there is no need to check Axiom 2 ($\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$) for W because it holds for all vectors in V and consequently for all vectors in W.
  - Other axioms inherited by W from V are 3, 7, 8, 9, and 10.
  - Only Axioms 1, 4, 5, 6 are needed to be checked.

Theorem 4.2.1

- Theorem 4.2.1
  - If W is a set of one or more vectors from a vector space V, then W is a subspace of V if and only if the following conditions hold:
    a) If **u** and **v** are vectors in W, then **u** + **v** is in W.
    b) If k is any scalar and **u** is any vector in W , then k**u** is in W.

Proof of Theorem 4.2.1

Axiom 1: If **u** and **v** are objects in *V*, then **u** + **v** is in *V*.

Axiom 6: If *k* is any scalar and **u** is any object in *V*, then *k***u** is in *V*.

- If *W* is a subspace of *V*, then all the vector space axioms are satisfied, including Axioms 1 and 6, which are precisely conditions (a) and (b).
- Conversely, assume conditions (a) and (b) hold. Since these conditions are vector space Axioms 1 and 6, we need only show that *W* satisfies the remaining eight axioms.

Proof of Theorem 4.2.1

- Axioms 2, 3, 7, 8, 9, and 10 are automatically satisfied by the vectors in *W* since they are satisfied by all vectors in *V*.
  - Therefore, we need only verify Axioms 4 and 5.

- Let **u** be any vector in *W*. By condition (b), *k***u** is in *W* for every scalar *k*.
- Setting $k=0$, $0\mathbf{u} = \mathbf{0}$ is in *W*, and setting $k=-1$, $(-1)\mathbf{u} = -\mathbf{u}$ is in *W* – Axioms 4 and 5 hold in *W*

Remark

- Theorem 4.2.1 states that ***W*** is a subspace of ***V*** if and only if ***W*** is a closed under addition (condition (a)) and closed under scalar multiplication (condition (b)).

Example

- Let W be any plane through the origin and let **u** and **v** be any vectors in W.
  - **u** + **v** must lie in W since it is the diagonal of the parallelogram determined by **u** and **v**, and k **u** must line in W for any scalar k since k **u** lies on a line through **u**.
- Thus, <u>W is closed under addition and scalar multiplication</u>, so it is a subspace of R^3.

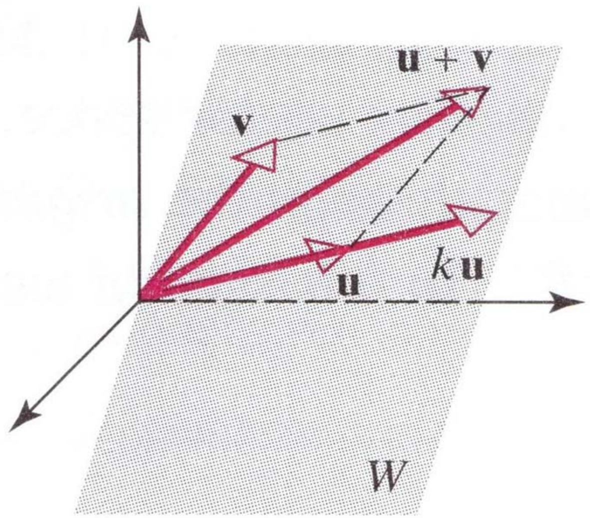
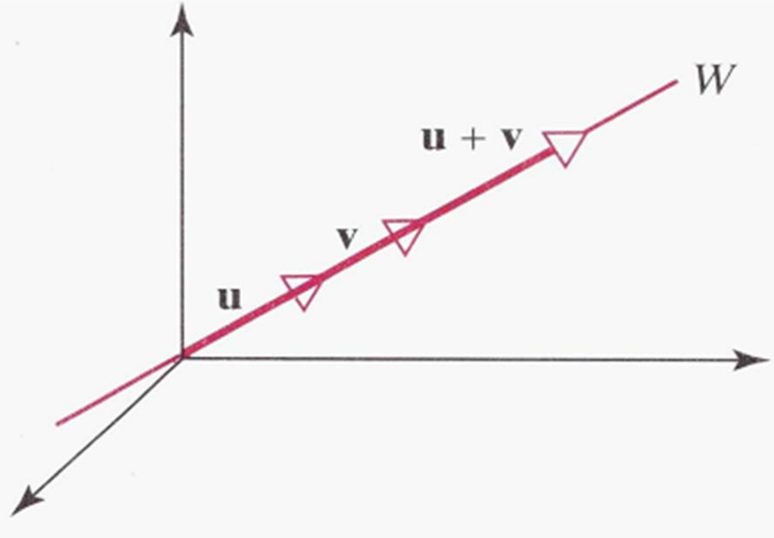


Figure 5.2.1
The vectors **u** + **v** and k**u** both lie in the same plane as **u** and **v**.

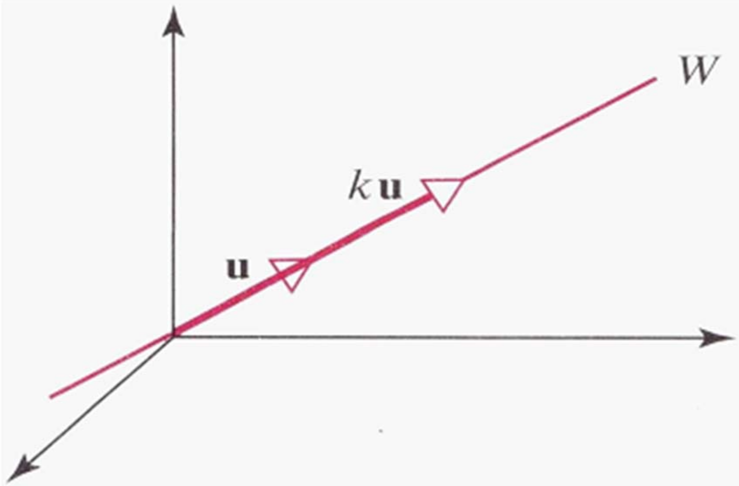
Example

- A line through the origin of R^3 is a subspace of R^3.

- Let W be a line through the origin of R^3.



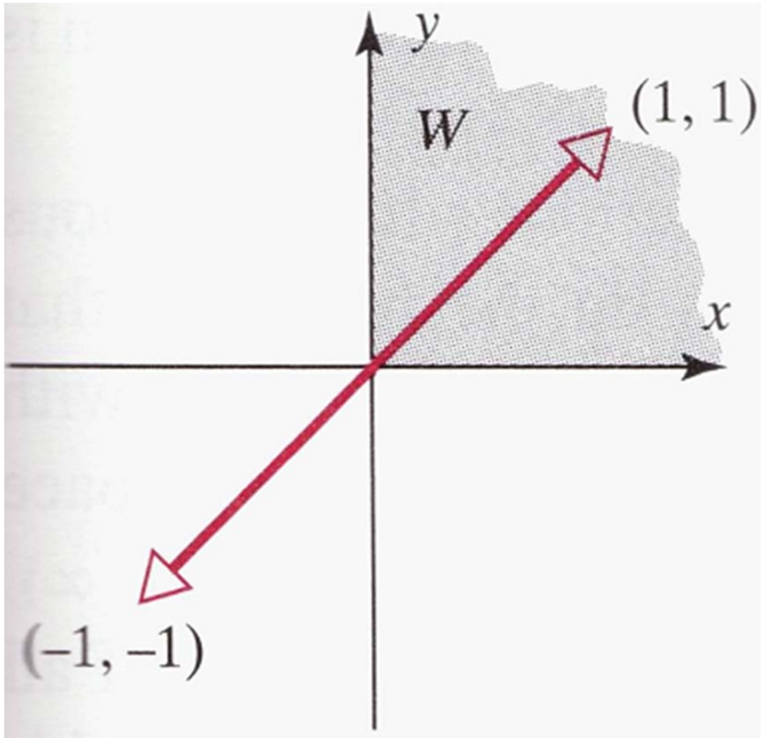
(a) W is closed under addition.



(b) W is closed under scalar multiplication.

Example (Not a Subspace)

- Let W be the set of all points (x, y) in R^2 such that $x \geq 0$ and $y \geq 0$. These are the points in the first quadrant.
- The set W is not a subspace of R^2 since it is not closed under scalar multiplication.
- For example, $\mathbf{v} = (1, 1)$ lines in W, but its negative $(-1)\mathbf{v} = -\mathbf{v} = (-1, -1)$ does not.



Remarks

Think about "set" and "empty set"!

- Every nonzero vector space V has at least two subspace: V itself is a subspace, and the set $\{\mathbf{0}\}$ consisting of just the zero vector in V is a subspace called the zero subspace.
- Examples of subspaces of R^2 and R^3:
  - Subspaces of R^2:
    - $\{\mathbf{0}\}$
    - Lines through the origin
    - R^2
  - Subspaces of R^3:
    - $\{\mathbf{0}\}$
    - Lines through the origin
    - Planes through origin
    - R^3
- They are actually the only subspaces of R^2 and R^3

Subspaces of M_{nn}

- Since the sum of two symmetric matrices is symmetric, and a scalar multiple of a symmetric matrix is symmetric. Thus, the set of $n \times n$ symmetric matrices is a subspace of the vector space M_{nn} of $n \times n$ matrices.
- Similarly, the set of $n \times n$ upper triangular matrices, the set of $n \times n$ lower triangular matrices, and the set of $n \times n$ diagonal matrices all form subspaces of M_{nn}, since each of these sets is closed under addition and scalar multiplication.

A Subset of M_{nn} That is Not a Subspace

- The set W of invertible $n \times n$ matrices is not a subspace of M_{nn}.

$$U = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \qquad V = \begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix}$$

- The matrix $0U$ is the 2×2 zero matrix and hence is not invertible – ***not closure under scalar multiplication***.
- The matrix $U+V$ has a column of zeros, so it is not invertible – ***not closure under addition***.

Theorem 4.2.2

- If $W_1, W_2, \ldots, W_r$ are subspaces of a vector space V, then the intersection of these subspaces is also a subspace of V.
- Proof:
- Let W be the intersection of the subspaces $W_1, W_2, \ldots, W_r$. It's not empty because each of these subspaces contains the zero vector of V.
- To prove closure under addition, let **u** and **v** be vectors in W. It follows that **u** and **v** also lie in each of these subspaces. Since these subspaces are all closed under addition, they all contain the vector **u**+**v** and hence so dos W.

Linear Combination

- Definition
  - A vector $\mathbf{w}$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ if it can be expressed in the form $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r$ where $k_1, k_2, \ldots, k_r$ are scalars.

- Vectors in R^3 are linear combinations of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$
  - Every vector $\mathbf{v} = (a, b, c)$ in R^3 is expressible as a linear combination of the standard basis vectors

$$\mathbf{i} = (1, 0, 0),\ \ \mathbf{j} = (0, 1, 0),\ \ \mathbf{k} = (0, 0, 1)$$

since

$$\mathbf{v} = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$$

Theorem 4.2.3

- If $S=\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space V, then:
  - The set W of all linear combinations of the vectors in S is a subspace of V.
  - W is the smallest subspace of V that contain all the vectors in S in the sense that any other subspace that contains those vectors contains W.

Proof of Theorem 4.2.3 (a)

- To show that W is a subspace of V, we must prove that it is closed under addition and scalar multiplication.
- There is at least one vector in W, namely **0**, since $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \ldots + 0\mathbf{v}_r$.
- If **u** and **v** are vectors in W, then

$$\mathbf{u} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \ldots + c_r\mathbf{w}_r$$

$$\mathbf{v} = k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \ldots + k_r\mathbf{w}_r$$

where $c_1, c_2, \ldots, c_r, k_1, k_2, \ldots, k_r$ are scalars.

$$\mathbf{u} + \mathbf{v} = (c_1+k_1)\mathbf{w}_1 + (c_2+k_2)\mathbf{w}_2 + \ldots + (c_r+k_r)\mathbf{w}_r$$

$$k\mathbf{u} = (kc_1)\mathbf{w}_1 + (kc_2)\mathbf{w}_2 + \ldots + (kc_r)\mathbf{w}_r$$

Proof of Theorem 4.2.3 (b)

- Each vector $\mathbf{w}_i$ is a linear combination of $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ since we can write

$$\mathbf{w}_i = 0\mathbf{w}_1 + 0\mathbf{w}_2 + \ldots + 1\mathbf{w}_i + \ldots + 0\mathbf{w}_r$$

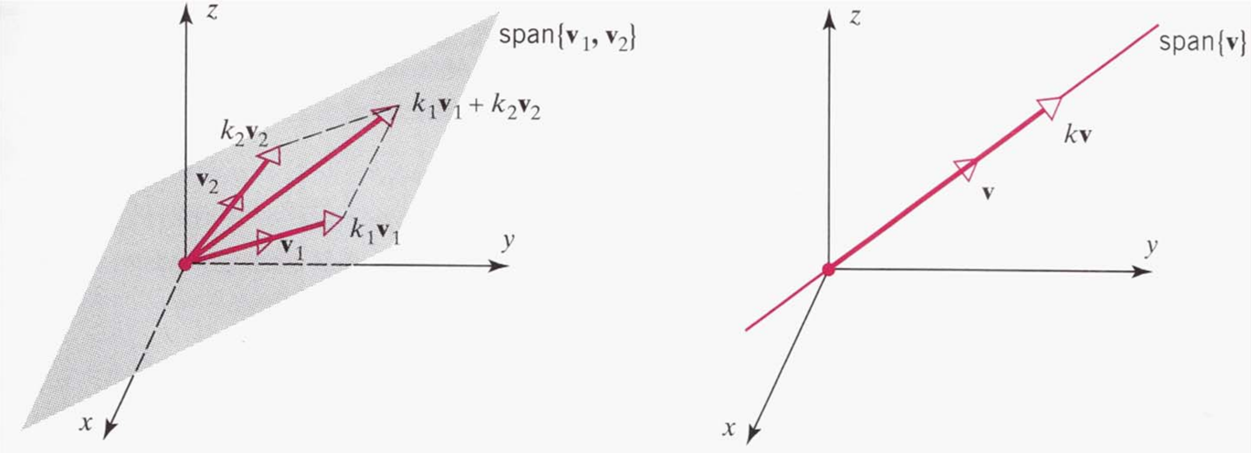
- Therefore, the subspace W contains each of the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$. Let W' be any other subspace that contains $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$. Since W' is closed under addition and scalar multiplication, it must contain all linear combination of $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$. Thus, W' contains each vector of W.

Span (展開)

- Definition
  - If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a set of vectors in a vector space V, then the subspace W of V containing of all linear combination of these vectors in S is called the space spanned by $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$, and we say that the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ span W.
  - To indicate that W is the space spanned by the vectors in the set $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$, we write $W = \text{span}(S)$ or $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$.

Example

- If **v** is a nonzero vector in R^2 and R^3, then span{**v**}, which is the set of all scalar multiples $k\mathbf{v}$, is the line determined by **v**.
- If $\mathbf{v}_1$ and $\mathbf{v}_2$ are non-collinear vectors in R^3 with their initial points at the origin, then span{$\mathbf{v}_1$, $\mathbf{v}_2$}, which consists of all linear combinations $k_1\mathbf{v}_1 + k_2\mathbf{v}_2$ is the plane determined by $\mathbf{v}_1$ and $\mathbf{v}_2$.



(a) Span{$\mathbf{v}_1$, $\mathbf{v}_2$} is the plane through the origin determined by $\mathbf{v}_1$ and $\mathbf{v}_2$.

(b) Span{**v**} is the line through the origin determined by **v**.

A Spanning Set for P_n

- The polynomials $1, x, x^2, \ldots, x^n$ span the vector space P_n since each polynomial **p** in P_n can be written as

$$\mathbf{p} = a_0 + a_1 x + \ldots + a_n x^n$$

which is a linear combination of $1, x, x^2, \ldots, x^n$

- We can denote this by writing

$$P_n = \text{span}\{1, x, x^2, \ldots, x^n\}$$

Example

Consider the vectors **u** = (1, 2, -1) and **v** = (6, 4, 2) in R^3. Show that **w** = (9, 2, 7) is a linear combination of **u** and **v** and that **w′** = (4, -1, 8) is not a linear combination of **u** and **v.**

Solution.

In order for **w** to be a linear combination of **u** and **v**, there must be scalars k_1 and k_2 such that $\mathbf{w} = k_1\mathbf{u} + k_2\mathbf{v}$;

$$(9, 2, 7) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

Equating corresponding components gives

$$\begin{aligned} k_1 + 6k_2 &= 9 \\ 2k_1 + 4k_2 &= 2 \\ -k_1 + 2k_2 &= 7 \end{aligned}$$

Solving this system yields $k_1 = -3$, $k_2 = 2$, so

$$\mathbf{w} = -3\mathbf{u} + 2\mathbf{v}$$

Example

Similarly, for **w**' to be a linear combination of **u** and **v**, there must be scalars k_1 and k_2 such that **w**'= k_1**u** + k_2**v**;

$$(4, -1, 8) = k_1(1, 2, -1) + k_2(6, 4, 2)$$

or

$$(4, -1, 8) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

Equating corresponding components gives

$$\begin{aligned} k_1 + 6k_2 &= 4 \\ 2k_1 + 4k_2 &= -1 \\ -k_1 + 2k_2 &= 8 \end{aligned}$$

This system of equation is inconsistent, so no such scalars k_1 and k_2 exist. Consequently, **w**' is not a linear combination of **u** and **v**.

Testing for Spanning

- Determine whether $\mathbf{v}_1$=(1,1,2), $\mathbf{v}_2$=(1,0,1), and $\mathbf{v}_3$=(2,1,3) span the vector space R^3.
- Solution: we must determine whether an arbitrary vector $\mathbf{b}=(b_1,b_2,b_3)$ in R^3 can be expressed as a linear combination $\mathbf{b}=k_1\mathbf{v}_1+k_2\mathbf{v}_2+k_3\mathbf{v}_3$.

$$(b_1, b_2, b_3) = k_1(1, 1, 2) + k_2(1, 0, 1) + k_3(2, 1, 3)$$

$$(b_1, b_2, b_3) = (k_1 + k_2 + 2k_3, k_1 + k_3, 2k_1 + k_2 + 3k_3)$$

$$\begin{aligned} &k_1 + k_2 + 2k_3 = b_1 \\ &k_1 + k_3 = b_2 \\ &2k_1 + k_2 + 3k_3 = b_3 \end{aligned}$$

Testing for Spanning

- This problem reduces to check whether this system is consistent
- Check the coefficient matrix

$$\begin{aligned} k_1 + k_2 + 2k_3 = b_1 \\ k_1 + k_3 = b_2 \\ 2k_1 + k_2 + 3k_3 = b_3 \end{aligned} \qquad A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

- The matrix A has the determinant equal to zero. This system is inconsistent. No solution can be found for k_1,k_2, and k_3. Therefore, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ do not span R^3.

Solution Space

- Solution Space of Homogeneous Systems
  - If $A\mathbf{x} = \mathbf{b}$ is a system of the linear equations, then each vector $\mathbf{x}$ that satisfies this equation is called a solution vector of the system.
  - Theorem 4.2.4 shows that the solution vectors of a homogeneous linear system form a vector space, which we shall call the solution space of the system.

- Theorem 4.2.4
  - If $A\mathbf{x} = \mathbf{0}$ is a homogeneous linear system of m equations in n unknowns, then the set of solution vectors is a subspace of R^n.

Proof of Theorem 4.2.4

- Let W be the set of solution vectors. There is at least one vector in W, namely $\mathbf{0}$.
- To show that W is closed under addition and scalar multiplication, we must show that if $\mathbf{x}$ and $\mathbf{x}'$ are any solution vectors and k is any scalar, then $\mathbf{x}+\mathbf{x}'$ and $k\mathbf{x}$ are also solution vectors.

$$A\mathbf{x} = \mathbf{0} \text{ and } A\mathbf{x}' = \mathbf{0}$$

$$A(\mathbf{x}+\mathbf{x}') = A\mathbf{x} + A\mathbf{x}' = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

$$A(k\mathbf{x}) = kA\mathbf{x} = k\mathbf{0} = \mathbf{0}$$

Which proves that $\mathbf{x}+\mathbf{x}'$ and $k\mathbf{x}$ are solution vectors.

Example

- Find the solution spaces of the linear systems.

$$(a)\begin{bmatrix}1 & -2 & 3\\ 2 & -4 & 6\\ 3 & -6 & 9\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \qquad (b)\begin{bmatrix}1 & -2 & 3\\ -3 & 7 & 8\\ -2 & 4 & -6\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$(c)\begin{bmatrix}1 & -2 & 3\\ -3 & 7 & -8\\ 4 & 1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \qquad (d)\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

- Each of these systems has three unknowns, so the solutions form subspaces of R^3.
- Geometrically, each solution space must be a line through the origin, a plane through the origin, the origin only, or all of R^3.

Example

Solution.

(a) $x = 2s - 3t, \quad y = s, \quad z = t$

$x = 2y - 3z$ or $x - 2y + 3z = 0$

This is the equation of the plane through the origin with **n** = (1, -2, 3) as a normal vector.

(b) $x = -5t, \quad y = -t, \quad z = t$

which are parametric equations for the line through the origin parallel to the vector **v** = (-5, -1, 1).

(c) The solution is x = 0, y = 0, z = 0, so the solution space is the origin only, that is {**0**}.

(d) The solution are x = r , y = s, z = t, where r, s, and t have arbitrary values, so the solution space is all of R^3.

Remark

- Whereas the solution set of every *homogeneous* system of *m* equations in *n* unknowns is a subspace of R^n, it is *never* true that the solution set of a *nonhomogeneous* system of *m* equations in *n* unknowns is a subspace of R^n.
- First, the system may not have any solutions at all
- Second, if there are solutions, then the solution set will not be closed under either addition or under scalar multiplication.

Remark

- Spanning sets are not unique. For example, any nonzero vector on the line will span that line.
- Theorem 4.2.5: If $S=\{\mathbf{v}_1,\mathbf{v}_2, \ldots, \mathbf{v}_r\}$ and $S'=\{\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_k\}$ are nonempty sets of vectors in a vector space V, then $span\{\mathbf{v}_1,\mathbf{v}_2, \ldots, \mathbf{v}_r\} = span\{\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_k\}$
if and only if each vector in S is a linear combination of those in S', and each vector in S' is a linear combination of those in S.

4.3
Linear Independence

Linearly Dependent & Independent

- Definition
  - If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a nonempty set of vector, then the vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r = \mathbf{0}$ has at least one solution, namely $k_1 = 0,\ k_2 = 0, \ldots, k_r = 0$. (trivial solution)
  - If this is the only solution, then S is called a linearly independent (線性獨立) set. If there are other solutions, then S is called a linearly dependent (線性相關) set.

- Examples
  - If $\mathbf{v}_1 = (2, -1, 0, 3)$, $\mathbf{v}_2 = (1, 2, 5, -1)$, and $\mathbf{v}_3 = (7, -1, 5, 8)$.
  - Then the set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, since $3\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$.

Example

- Let $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$ in R^3.
  - Consider the equation $k_1\mathbf{i} + k_2\mathbf{j} + k_3\mathbf{k} = 0$
    $\Rightarrow k_1(1, 0, 0) + k_2(0, 1, 0) + k_3(0, 0, 1) = (0, 0, 0)$
    $\Rightarrow (k_1, k_2, k_3) = (0, 0, 0)$
    $\Rightarrow$ The set $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is linearly independent.
- Similarly the vectors

  $\mathbf{e}_1 = (1, 0, 0, \ldots,0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $\mathbf{e}_n = (0, 0, 0, \ldots, 1)$

  form a linearly independent set in R^n.

- Remark:
  - To check whether a set of vectors is linear independent or not, write down the linear combination of the vectors and see if their coefficients all equal zero.

Example

- Determine whether the vectors

  $$\mathbf{v}_1 = (1, -2, 3),\ \mathbf{v}_2 = (5, 6, -1),\ \mathbf{v}_3 = (3, 2, 1)$$

  form a linearly dependent set or a linearly independent set.

- Solution
  - Let the vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = 0$

    $\Rightarrow k_1(1, -2, 3) + k_2(5, 6, -1) + k_3(3, 2, 1) = (0, 0, 0)$

    $$\Rightarrow \begin{aligned} k_1 + 5k_2 + 3k_3 &= 0 \\ -2k_1 + 6k_2 + 2k_3 &= 0 \\ 3k_1 - k_2 + k_3 &= 0 \end{aligned}$$

    $\Rightarrow \det(A) = 0$

    $\Rightarrow$ The system has nontrivial solutions

    $\Rightarrow \mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ form a linearly dependent set

> $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\Longleftrightarrow \det(A) \neq 0$

Example

- Determine whether the vectors $\mathbf{v}_1$=(1,2,2,-1), $\mathbf{v}_2$=(4,9,9,-4), $\mathbf{v}_3$=(5,8,9,-5) in R^4 are linearly independent or not.
- Solution:
  - Check $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$.
  - $k_1(1,2,2,-1) + k_2(4,9,9,-4) + k_3(5,8,9,-5) = (0,0,0,0)$

$$\begin{aligned} k_1 + 4k_2 + 5k_3 &= 0 \\ 2k_1 + 9k_2 + 8k_3 &= 0 \\ 2k_1 + 9k_2 + 9k_3 &= 0 \\ -k_1 - 4k_2 - 5k_3 &= 0 \end{aligned}$$

  - The coefficient matrix is invertible. Thus this system has only the trivial solution, and $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent.

Theorem 4.3.1

- Theorem 4.3.1
  - A set *S* with two or more vectors is:
    - Linearly dependent if and only if at least one of the vectors in *S* is expressible as a linear combination of the other vectors in *S*.
    - Linearly independent if and only if no vector in *S* is expressible as a linear combination of the other vectors in *S*.

Proof of Theorem 4.3.1

- Let $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_r\}$ be a set with two or more vectors. If we assume that S is **linearly dependent**, then there are scalars k_1, k_2, …, k_r, not all zero, such that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r = \mathbf{0}$$

- To be specific, suppose that $k_1 \neq 0$. Then it can be rewritten as

$$\boldsymbol{v}_1 = \left(-\frac{k_2}{k_1}\right)\boldsymbol{v}_2 + \cdots + \left(-\frac{k_r}{k_1}\right)\boldsymbol{v}_r$$

  which expresses $\mathbf{v}_1$ as a linear combination of the other vectors in S.

- Similarly, if $k_j \neq 0$ for some $j=2,3,\ldots,r$, then $\mathbf{v}_j$ is expressible as a linear combination of the other vectors in S.

Proof of Theorem 4.3.1

- Conversely, let us assume that at least one of the vectors in S **is expressible as a linear combination of the other vectors**. To be specific, suppose that

$$\mathbf{v}_1 = c_2\mathbf{v}_2+c_3\mathbf{v}_3+\ldots+c_r\mathbf{v}_r$$

- So $\mathbf{v}_1 - c_2\mathbf{v}_2 - c_3\mathbf{v}_3 - \ldots - c_r\mathbf{v}_r = \mathbf{0}$
- If follows that S is **linearly dependent** since the equation

$$k_1\mathbf{v}_1+k_2\mathbf{v}_2+\ldots+k_r\mathbf{v}_r = \mathbf{0}$$

is satisfied by $k_1=1$, $k_2=-c_2$, …, $k_r=-c_r$

which are not all zero. The proof in the case where some vector other than $\mathbf{v}_1$ is expressible as a linear combination of the other vectors in S is similar.

Example

- The vectors $\mathbf{v}_1$ = (2,-1,0,3), $\mathbf{v}_2$=(1,2,5,-1), and $\mathbf{v}_3$=(7,-1,5,8)
- From Theorem 4.3.1, at least one of these vectors is expressible as a linear combination of the other two.
- $3\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$.

$$\boldsymbol{v}_1 = -\tfrac{1}{3}\boldsymbol{v}_2 + \tfrac{1}{3}\boldsymbol{v}_3 \qquad \boldsymbol{v}_2 = -3\boldsymbol{v}_1 + \boldsymbol{v}_3 \qquad \boldsymbol{v}_3 = 3\boldsymbol{v}_1 + \boldsymbol{v}_2$$

Example

- The vectors **i**=(1,0,0), **j**=(0,1,0), and **k**=(0,0,1)
- Suppose that **k** is expressible as $\mathbf{k} = k_1\mathbf{i} + k_2\mathbf{j}$
- Then, in terms of components,

$$(0,0,1) = k_1(1,0,0) + k_2(0,1,0)$$

$$(0,0,1) = (k_1,k_2,0)$$

- The last equation is not satisfied by any values of k_1 and k_2, so **k** cannot be expressed as a linear combination of **i** and **j**.

Theorem 4.3.2

- Theorem 4.3.2
  - A finite set of vectors that contains the zero vector **0** is linearly dependent.
  - A set with exactly one vector is linearly independent if and only if that vector is not **0**.
  - A set with exactly two vectors is linearly independently if and only if neither vector is a scalar multiple of the other.

Proof of Theorem 4.3.2(a)

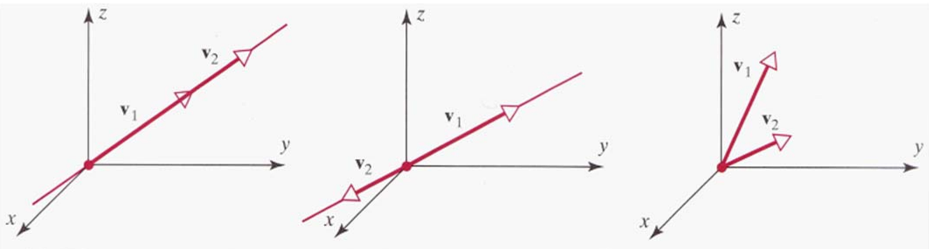
- For any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$, the set $S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{0}\}$ is linearly dependent since the equation

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \ldots + 0\mathbf{v}_r + 1(\mathbf{0}) = \mathbf{0}$$

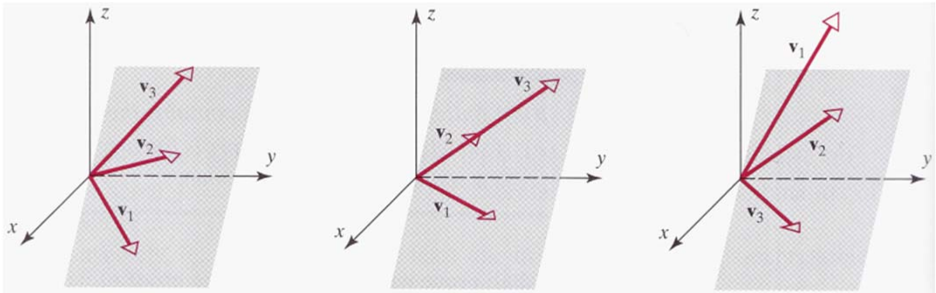
expresses $\mathbf{0}$ as a linear combination of the vectors in S with coefficients that are not all zero.

Geometric Interpretation of Linear Independence

- In R^2 and R^3, a set of two vectors is linearly independent if and only if the vectors do not lie on the same line when they are placed with their initial points at the origin.
- In R^3, a set of three vectors is linearly independent if and only if the vectors do not lie in the same plane when they are placed with their initial points at the origin.



(a) Linearly dependent (b) Linearly dependent (c) Linearly independent



(a) Linearly dependent (b) Linearly dependent (c) Linearly independent

Theorem 4.3.3

- Theorem 4.3.3
  - Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of vectors in R^n. If $r > n$, then S is linearly dependent.
- Proof
  - Suppose that

$$\mathbf{v}_1 = (v_{11}, v_{12}, \ldots, v_{1n})$$
$$\mathbf{v}_2 = (v_{21}, v_{22}, \ldots, v_{2n})$$
$$\ldots$$
$$\mathbf{v}_r = (v_{r1}, v_{r2}, \ldots, v_{rn})$$

Proof of Theorem 4.3.3

- Consider the equation $k_1\mathbf{v}_1+k_2\mathbf{v}_2+\ldots+k_r\mathbf{v}_r=\mathbf{0}$.
- We express both sides of this equation in terms of components and then equate corresponding components, we obtain the system

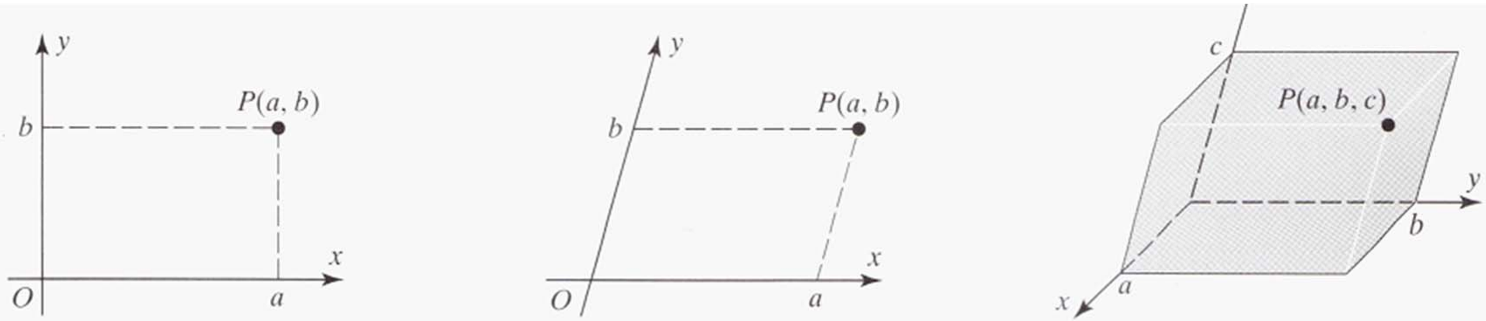
$$\begin{aligned} v_{11}k_1 + v_{21}k_2 + \cdots + v_{r1}k_r &= 0 \\ v_{12}k_1 + v_{22}k_2 + \cdots + v_{r2}k_r &= 0 \\ \vdots \qquad \vdots \qquad \vdots \qquad & \\ v_{1n}k_1 + v_{2n}k_2 + \cdots + v_{rn}k_r &= 0 \end{aligned}$$

- This is a homogeneous system of n equations in the r unknowns $k_1, \ldots, k_r$. Since $r > n$, it follows from Theorem 1.2.2 that the system has nontrivial solutions. Therefore, S is a linearly dependent set.

4.4
Coordinates and Basis

Nonrectangular Coordinate Systems

- The coordinate system establishes a one-to-one correspondence between points in the plane and ordered pairs of real numbers.
- Although perpendicular coordinate axes are the most common, any two nonparallel lines can be used to define a coordinate system in the plane.



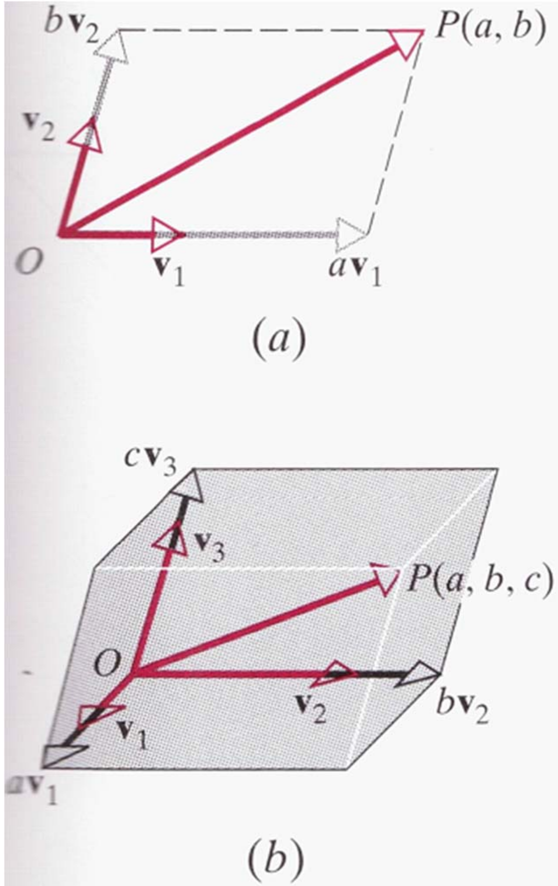
(a) Coordinates of P in a rectangular coordinate system in 2-space

(b) Coordinates of P in a nonrectangular coordinate system in 2-space

(c) Coordinates of P in a nonrectangular coordinate system in 3-space

Nonrectangular Coordinate Systems

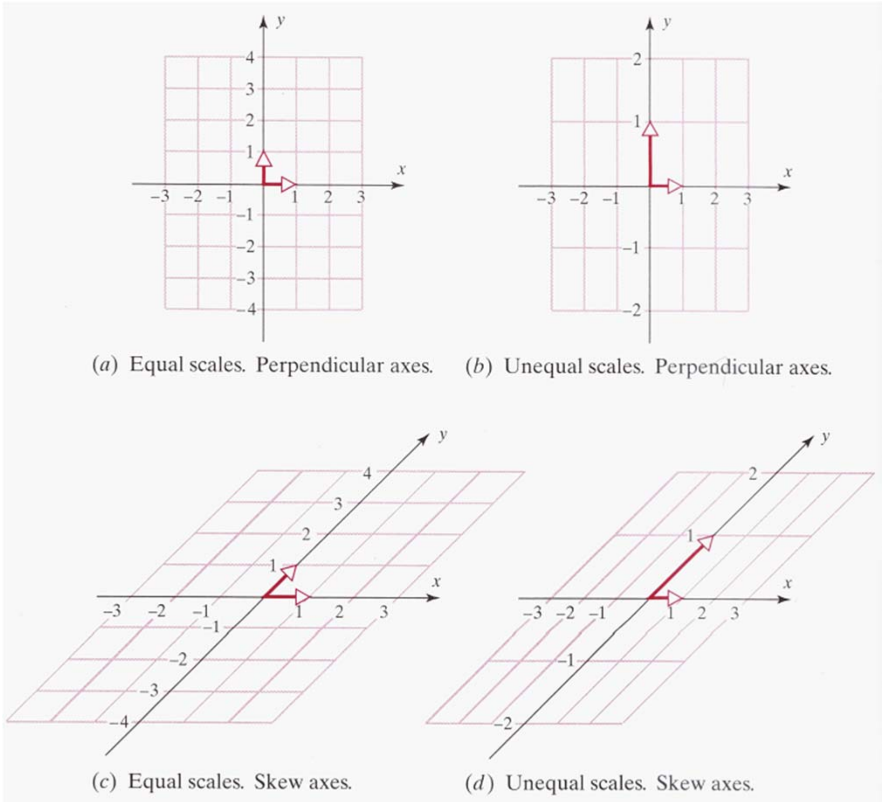
- A coordinate system can be constructed by general vectors:
  - $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors of length 1 that points in the positive direction of the axis: $\overrightarrow{OP} = a\mathbf{v}_1 + b\mathbf{v}_2$
  - Similarly, the coordinates (a, b, c) of the point P can be obtained by expressing $\overrightarrow{OP}$ as a linear combination of the vectors



Nonrectangular Coordinate Systems

- Informally stated, vectors that specify a coordinate system are called “basis vectors” for that system.

- Although we used basis vectors of length 1 in the preceding discussion, this is not essential – nonzero vectors of any length will suffice.



Basis

- Definition
  - If V is any vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set of vectors in V, then S is called a basis (基底) for V if the following two conditions hold:
    - S is linearly independent.
    - S spans V.

Example

- The standard basis for R^n

  $\mathbf{e}_1=(1,0,\ldots,0)$, $\mathbf{e}_2 = (0,1,0,\ldots,0)$, …, $\mathbf{e}_n = (0,0,\ldots,1)$

- Show that $\mathbf{v}_1=(1,2,1)$, $\mathbf{v}_2=(2,9,0)$, $\mathbf{v}_3=(3,3,4)$ form a basis for R^3
  - Solution:
  - check that $c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3 = 0$ has only trivial solution. (linearly independent)
  - Check every vector $\mathbf{b}=(b_1,b_2,b_3)$ can be expressed as $c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3 = \mathbf{b}$ (span)

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 0 \\ 2c_1 + 9c_2 + 3c_3 &= 0 \\ c_1 + 4c_3 &= 0 \end{aligned} \qquad \begin{aligned} c_1 + 2c_2 + 3c_3 &= b_1 \\ 2c_1 + 9c_2 + 3c_3 &= b_2 \\ c_1 + 4c_3 &= b_3 \end{aligned}$$

  - Because $\det(A) \neq 0$, $\mathbf{v}_1$,$\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for R^3. (Theorem 2.3.8)

Definition

- A vector space that cannot be spanned by finitely many vectors is said to be ***infinite-dimensional***, whereas those that can are said to be ***finite-dimensional***.

Theorem 4.4.1

- Theorem 4.4.1 (Uniqueness of Basis Representation)
  - If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

    in **exactly one way**.

Proof of Theorem 4.4.1

- Since *S* spans *V*, it follows from the definition of a spanning set that every vector in *V* is expressible as a linear combination of the vectors in *S*.
- To see that there is only one way to express a vector as a linear combination of the vectors in *S*, suppose that some vector **v** can be written as

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

- And also as

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n$$

Proof of Theorem 4.4.1

- Subtracting the second equation from the first gives

$$\mathbf{0} = (c_1\text{-}k_1)\mathbf{v}_1+(c_2\text{-}k_2)\mathbf{v}_2+\ldots+(c_n\text{-}k_n)\mathbf{v}_n$$

- Since the right side is a linear combination of vectors in S, the linear independent of S implies that

$$c_1 - k_1 = 0,\ c_2 - k_2 = 0,\ \ldots,\ c_n - k_n = 0$$

- That is $c_1=k_1$, $c_2=k_2$, …, $c_n=k_n$.
- Thus, the two expressions for **v** are the same.

Coordinates Relative to a Basis

- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, and

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

  is the expression for a vector **v** in terms of the basis S, then the scalars c_1, c_2, …, c_n, are called the coordinates (座標) of **v** relative to the basis S.
- The vector $(c_1, c_2, \ldots, c_n)$ in R^n constructed from these coordinates is called the coordinate vector of **v** relative to S (v對於基底S的座標向量); it is denoted by

$$(\mathbf{v})_S = (c_1, c_2, \ldots, c_n)$$

- Remark:
  - Coordinate vectors depend not only on the basis S but also on the order in which the basis vectors are written.
  - A change in the order of the basis vectors results in a corresponding change of order for the entries in the coordinate vector.

Example (Standard Basis for R^3)

- Suppose that $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$, then $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a linearly independent set in R^3.

- This set also spans R^3 since any vector $\mathbf{v} = (a, b, c)$ in R^3 can be written as

$$\mathbf{v} = (a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

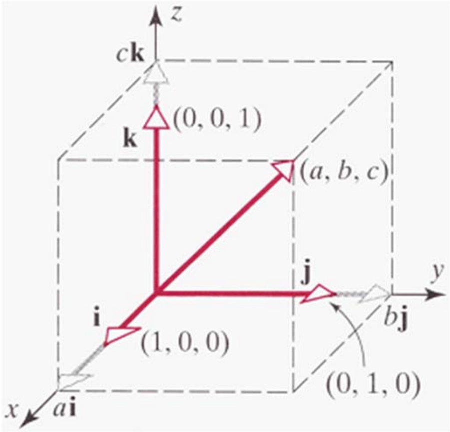
- Thus, S is a basis for R^3; it is called the standard basis for R^3.

- Looking at the coefficients of **i**, **j**, and **k**, it follows that the coordinates of **v** relative to the standard basis are a, b, and c, so

$$(\mathbf{v})_S = (a, b, c)$$

- Comparing this result to $\mathbf{v} = (a, b, c)$, we have

$$\mathbf{v} = (\mathbf{v})_S$$



Example (Representing a Vector Using Two Bases)

$\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 9, 0)$, and $\mathbf{v}_3 = (3, 3, 4)$

- Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be the basis for R^3 in the preceding example.
  - Find the coordinate vector of $\mathbf{v} = (5, -1, 9)$ with respect to S.
  - Find the vector $\mathbf{v}$ in R^3 whose coordinate vector with respect to the basis S is $(\mathbf{v})_s = (-1, 3, 2)$.

- Solution (a)
  - We must find scalars c_1, c_2, c_3 such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, or, in terms of components, $(5, -1, 9) = c_1(1, 2, 1) + c_2(2, 9, 0) + c_3(3, 3, 4)$
  - Solving this, we obtaining $c_1 = 1$, $c_2 = -1$, $c_3 = 2$.
  - Therefore, $(\mathbf{v})_s = (1, -1, 2)$.
- Solution (b)
  - Using the definition of the coordinate vector $(\mathbf{v})_S$, we obtain

$$\mathbf{v} = (-1)\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_3 = (11, 31, 7).$$

Standard Basis for P_n

- $S = \{1, x, x^2, \ldots, x^n\}$ is a basis for the vector space P_n of polynomials of the form $a_0 + a_1x + \ldots + a_nx^n$. The set S is called the standard basis for P_n.
  Find the coordinate vector of the polynomial $\mathbf{p} = a_0 + a_1x + a_2x^2$ relative to the basis $S = \{1, x, x^2\}$ for P_2 .

- Solution:
  - The coordinates of $\mathbf{p} = a_0 + a_1x + a_2x^2$ are the scalar coefficients of the basis vectors 1, x, and x^2, so

$$(\mathbf{p})_S = (a_0, a_1, a_2).$$

Standard Basis for M_{mn}

- Let $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
- The set $S = \{M_1, M_2, M_3, M_4\}$ is a basis for the vector space M_{22} of 2×2 matrices.
- To see that S spans M_{22}, note that an arbitrary vector (matrix) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be written as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = aM_1 + bM_2 + cM_3 + dM_4$$

- To see that S is linearly independent, assume $aM_1 + bM_2 + cM_3 + dM_4 = 0$. It follows that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Thus, $a = b = c = d = 0$, so S is lin. indep.
- The basis S is called the standard basis for M_{22}.
- More generally, the standard basis for M_{mn} consists of the mn different matrices with a single 1 and zeros for the remaining entries.

4.5 Dimension

Finite-Dimensional

- Definition
  - A nonzero vector space V is called finite-dimensional (有限維的) if it contains a finite set of vector $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ that forms a basis. If no such set exists, V is called infinite-dimensional (無限維的). In addition, we shall regard the zero vector space to be finite-dimensional.

- Example
  - The vector space R^n is finite-dimensional.

Theorems

- Theorem 4.5.1
  - All bases for a finite-dimensional vector space have the same number of vectors.
- Theorem 4.5.2
  - Let V be a finite-dimensional vector space and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ any basis.
    - If a set has more than n vector, then it is linearly dependent.
    - If a set has fewer than n vector, then it does not span V.

Proof of Theorem 4.5.2(a)

If a set has more than *n* vector, then it is linearly dependent.

- Let $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ be any set of *m* vectors in *V*, where $m > n$. Since $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis, each $\mathbf{w}_i$ can be expressed as a linear combination of the vectors in *S*, say

$$
\begin{aligned}
\boldsymbol{w}_1 &= a_{11}\boldsymbol{v}_1 + a_{21}\boldsymbol{v}_2 + \cdots + a_{n1}\boldsymbol{v}_n \\
\boldsymbol{w}_2 &= a_{12}\boldsymbol{v}_1 + a_{22}\boldsymbol{v}_2 + \cdots + a_{n2}\boldsymbol{v}_n \\
\vdots \quad & \vdots \\
\boldsymbol{w}_m &= a_{1m}\boldsymbol{v}_1 + a_{2m}\boldsymbol{v}_2 + \cdots + a_{nm}\boldsymbol{v}_n
\end{aligned}
$$

- To show that *S'* is linearly dependent, we must find scalars $k_1, k_2, \ldots, k_m$, not all zero, such that

$$k_1\boldsymbol{w}_1 + k_2\boldsymbol{w}_2 + \cdots + k_m\boldsymbol{w}_m = \mathbf{0}$$

Proof of Theorem 4.5.2(a)

If a set has more than *n* vector, then it is linearly dependent.

- Using the previous results, we can rewrite

$$
\begin{aligned}
&(k_1a_{11} + k_2a_{12} + \cdots + k_ma_{1m})\boldsymbol{v}_1 \\
&+ (k_1a_{21} + k_2a_{22} + \cdots + k_ma_{2m})\boldsymbol{v}_2 \\
&\vdots \\
&+ (k_1a_{n1} + k_2a_{n2} + \cdots + k_ma_{nm})\boldsymbol{v}_n = \mathbf{0}
\end{aligned}
$$

- Thus, from the linear independence of *S*, the problem of proving that *S*' is a linearly dependent set reduces to showing there are scalars $k_1, k_2, \ldots, k_m$, not all zero, that satisfy

$$
\begin{aligned}
a_{11}k_1 + a_{12}k_2 + \cdots + a_{1m}k_m &= 0 \\
a_{21}k_1 + a_{22}k_2 + \cdots + a_{2m}k_m &= 0 \\
a_{n1}k_1 + a_{n2}k_2 + \cdots + a_{nm}k_m &= 0
\end{aligned}
$$

Proof of Theorem 4.5.2(a)

If a set has more than *n* vector, then it is linearly dependent.

$$
\begin{aligned}
a_{11}k_1 + a_{12}k_2 + \cdots + a_{1m}k_m &= 0 \\
a_{21}k_1 + a_{22}k_2 + \cdots + a_{2m}k_m &= 0 \\
a_{n1}k_1 + a_{n2}k_2 + \cdots + a_{nm}k_m &= 0
\end{aligned}
$$

- It has more unknowns than equations ($m > n$), so the proof is complete since Theorem 1.2.2 guarantees the existence of nontrivial solutions.

Proof of Theorem 4.5.2(b)

If a set has fewer than *n* vector, then it does not span *V*.

- Let $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ be any set of *m* vectors in *V*, where $m < n$. The proof will be by contradiction: We will show that assuming *S*' span *V* leads to a contradiction of the linear independence of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.
- If *S*' spans *V*, then every vector in *V* is a linear combination of the vectors in *S*'. In particular, each basis vectors $\mathbf{v}_i$ is a linear combination of the vectors in *S*', say

$$
\begin{aligned}
\boldsymbol{v}_1 &= a_{11}\boldsymbol{w}_1 + a_{21}\boldsymbol{w}_2 + \cdots + a_{m1}\boldsymbol{w}_m \\
\boldsymbol{v}_2 &= a_{12}\boldsymbol{w}_1 + a_{22}\boldsymbol{w}_2 + \cdots + a_{m2}\boldsymbol{w}_m \\
&\vdots \\
\boldsymbol{v}_n &= a_{1n}\boldsymbol{w}_1 + a_{2n}\boldsymbol{w}_2 + \cdots + a_{mn}\boldsymbol{w}_m
\end{aligned}
$$

Proof of Theorem 4.5.2(b)

If a set has fewer than *n* vector, then it does not span *V*.

- To obtain our contradiction, we will show that there are scalars $k_1,k_2,\ldots,k_n$, not all zero, such that

$$k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_n\boldsymbol{v}_n = \mathbf{0}$$

- Thus the computation now yield

$$\begin{aligned} a_{11}k_1 + a_{12}k_2 + \cdots + a_{1n}k_n &= 0 \\ a_{21}k_1 + a_{22}k_2 + \cdots + a_{2n}k_n &= 0 \\ a_{m1}k_1 + a_{m2}k_2 + \cdots + a_{mn}k_n &= 0 \end{aligned}$$

- This linear system has more unknowns than equations ($m < n$) and hence has nontrivial solution by Theorem 1.2.2.

Dimension

- Definition
  - The dimension (維度) of a finite-dimensional vector space V, denoted by $\dim(V)$, is defined to be the number of vectors in a basis for V.
  - We define the zero vector space to have dimension zero.

- Dimensions of Some Vector Spaces:
  - $\dim(R^n) = n$ [The standard basis has n vectors]

Example

- Determine a basis and the dimension of the solution space of the homogeneous system

$$\begin{aligned} 2x_1 + 2x_2 - x_3 + x_5 &= 0 \\ -x_1 + x_2 + 2x_3 - 3x_4 + x_5 &= 0 \\ x_1 + x_2 - 2x_3 - x_5 &= 0 \\ x_3 + x_4 + x_5 &= 0 \end{aligned}$$

- Solution:
  - The general solution of the given system is

    $x_1 = -s-t,\ \ x_2 = s,$

    $x_3 = -t,\ \ x_4 = 0,\ \ x_5 = t$
  - Therefore, the solution vectors can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

  - Which shows that the vectors

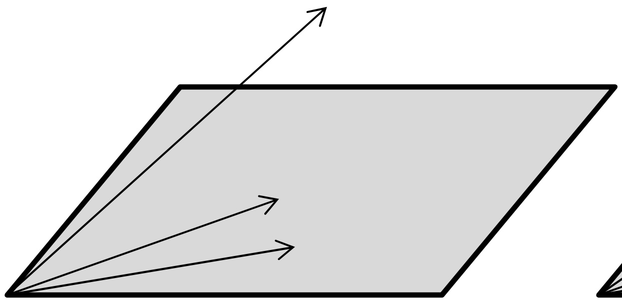
$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

    span the solution space.
  - Since they are also linearly independent, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis , and the solution space is two-dimensional.

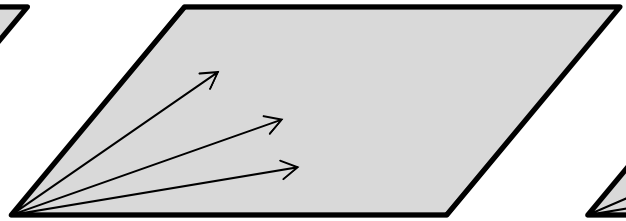
Theorem 4.5.3

- Theorem 4.5.3 (Plus/Minus Theorem)
  - Let S be a nonempty set of vectors in a vector space V.
    - If S is a linearly independent set, and if $\mathbf{v}$ is a vector in V that is outside of span(S), then the set $S \cup \{\mathbf{v}\}$ that results by inserting $\mathbf{v}$ into S is still linearly independent.
    - If $\mathbf{v}$ is a vector in S that is expressible as a linear combination of other vectors in S, and if $S - \{\mathbf{v}\}$ denotes the set obtained by removing $\mathbf{v}$ from S, then S and $S - \{\mathbf{v}\}$ span the same space; that is, $\text{span}(S) = \text{span}(S - \{\mathbf{v}\})$

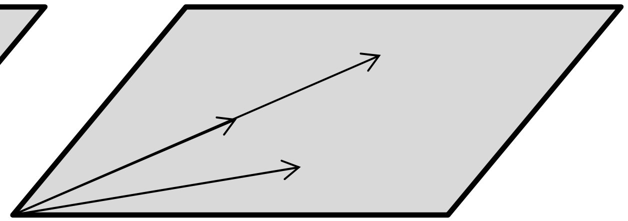
Examples



None of the three vectors lies in the same plane as the other two.



Any of the vectors can be removed, and the remaining two will still span the plane.



Either of the collinear vectors can be removed, and the remaining two will still span the plane.

Theorem 4.5.4

- Theorem 4.5.4
  - If V is an n-dimensional vector space, and if S is a set in V with exactly n vectors, then S is a basis for V if either S spans V or S is linearly independent.

Proof of Theorem 4.5.4

- Assume that S has exactly n vectors and spans V. To prove that S is a basis, we must show that S is a linearly independent set. But if this is not so, then some vector $\mathbf{v}$ in S is a linear combination of the remaining vectors.
- If we remove this vector from S, then it follows from the Plus/Minus Theorem that the remaining set of n-1 vectors still spans V. But this is impossible, since it follows from Theorem 4.5.2*b* that no set with fewer than n vectors can span an n-dimensional vector space. Thus S is linearly independent.

Proof of Theorem 4.5.4

- Assume that S has exactly n vectors and is a linearly independent set. To prove that S is a basis, we must show that S spans V.
- But if this is not so, then there is some vector $\mathbf{v}$ in V that is not in span(S). If we insert this vector into S, then it follows from the Plus/Minus Theorem that this set of $n+1$ vectors is still linearly independent.
- But this is impossible, since it follows from Theorem 4.5.2*a* that no set with more than n vectors in an n-dimensional vector space can be linearly independent. Thus S spans V.

Example

- Show that $\mathbf{v}_1 = (-3, 7)$ and $\mathbf{v}_2 = (5, 5)$ form a basis for R^2 by inspection.
- Solution:
  - Neither vector is a scalar multiple of the other
    $\Rightarrow$ The two vectors form a linear independent set in the 2-D space R^2
    $\Rightarrow$ The two vectors form a basis by Theorem 4.5.4.

- Show that $\mathbf{v}_1 = (2, 0, 1)$, $\mathbf{v}_2 = (4, 0, 7)$, $\mathbf{v}_3 = (-1, 1, 4)$ form a basis for R^3 by inspection.
- Solution:
  - The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a linearly independent set in the *xz*-plane.
  - The vector $\mathbf{v}_3$ is outside of the *xz*-plane, so the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also linearly independent.
  - Since R^3 is three-dimensional, Theorem 4.5.4 implies that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for R^3.

Theorem 4.5.5

- Theorem 4.5.5
  - Let S be a finite set of vectors in a finite-dimensional vector space V.
    - If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.
    - If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

Proof of Theorem 4.5.5(a)

- If *S* is a set of vectors that spans *V* but is not a basis for *V*, then *S* is a **linearly dependent** set. Thus some vector **v** in *S* is expressible as a linear combination of the other vectors in *S*.
- By the Plus/Minus Theorem, we can remove **v** from *S*, and the resulting set *S*' will still span *V*. If *S*' is linearly independent, then *S*' is a basis for *V*, and we are done.
- If *S*' is linearly dependent, then we can remove some appropriate vector from *S*' to produce a set *S*'' that still spans *V*.
- We can continue removing vectors until we finally arrive at a set of vectors in *S* that is linearly independent and spans *V*.

Proof of Theorem 4.5.5(b)

- Suppose that dim(*V*)=*n*. If *S* is a linearly independent set that is not already a basis for *V*, then *S* fails to span *V*, and there is some vector **v** in *V* that is not is span(*S*).
- By the Plus/Minus Theorem, we can insert **v** into *S*, and the resulting set *S*' will still be linearly independent. If *S*' spans *V*, then *S*' is a basis for *V*, and we are done.
- If *S*' does not span *V*, then we can insert an appropriate vector into *S*' to produce a set *S*'' that is still linearly independent.
- We can continue inserting vectors until we reach a set with *n* linearly independent vectors in *V*. This set will be a basis for *V*.

Theorem 4.5.6

- Theorem 4.5.6
  - If W is a subspace of a finite-dimensional vector space V, then W is finite-dimensional and $\dim(W) \leq \dim(V)$.
  - If $\dim(W) = \dim(V)$, then $W = V$.

Proof of Theorem 4.5.6

- Since V is finite-dimensional, so is W. Accordingly, suppose that $S=\{\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_m\}$ is a basis for W. Either S is also a basis for V or it is not.
- If it is, then $\dim(W)=\dim(V)=m$. If it is not, then by Theorem 4.5.5*b*, vectors can be added to the linearly independent set S to make it into a basis for V, so $\dim(W)<\dim(V)$. Thus $\dim(W) \leqq \dim(V)$ in all cases.
- If $\dim(W)=\dim(V)$, then S is a set of m linearly independent vectors in the m-dimensional vector space V; hence S is a basis for V by Theorem 4.5.5. This implies that $W = V$.

4.6
Change of Basis

Coordinate Maps

- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then each vector $\mathbf{v}$ in V can be expressed uniquely as a linear combination of the basis vectors, say

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_n\mathbf{v}_n$$

- The scalars $k_1, k_2, \ldots, k_n$ are the coordinates of $\mathbf{v}$ relative to S, and the vector

$$(\mathbf{v})_S = (k_1, k_2, \ldots, k_n)$$

  is the coordinate vector of $\mathbf{v}$ relative to S.
- Thus, we define

$$[\mathbf{v}]_S = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$$

  to be the coordinate vector of $\mathbf{v}$ relative to S.

Change of Basis (基底變換)

- Change of basis problem
  - If we change the basis for a vector space V from some old basis B to some new basis B', how is the old coordinate matrix $[\mathbf{v}]_B$ of a vector $\mathbf{v}$ related to the new coordinate matrix $[\mathbf{v}]_{B'}$?
- For simplicity, we show the example in 2-dimensional spaces. Let $B=\{\mathbf{u}_1,\mathbf{u}_2\}$ and $B'=\{\mathbf{u}_1',\mathbf{u}_2'\}$. We will need the coordinate vectors for the new basis vectors relative to the old basis. Suppose they are

$$[\boldsymbol{u}_1']_B = \begin{bmatrix} a \\ b \end{bmatrix} \qquad [\boldsymbol{u}_2']_B = \begin{bmatrix} c \\ d \end{bmatrix}$$

Change of Basis

- That is, $$\boldsymbol{u}_1' = a\boldsymbol{u}_1 + b\boldsymbol{u}_2$$ $$\boldsymbol{u}_2' = c\boldsymbol{u}_1 + d\boldsymbol{u}_2$$
- Now let **v** be any vector in *V*, and let $[\boldsymbol{v}]_{B'} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$ be the new coordinate vector, so that
  $\mathbf{v}=k_1\mathbf{u}_1'+k_2\mathbf{u}_2'$
- In order to find the old coordinates of **v**, we must express **v** in terms of the old basis *B*. This yields
  $$\boldsymbol{v} = k_1(a\boldsymbol{u}_1 + b\boldsymbol{u}_2) + k_2(c\boldsymbol{u}_1 + d\boldsymbol{u}_2)$$
  $$\boldsymbol{v} = (k_1a + k_2c)\boldsymbol{u}_1 + (k_1b + k_2d)\boldsymbol{u}_2$$

Change of Basis

$$\boldsymbol{v} = (k_1 a + k_2 c)\boldsymbol{u}_1 + (k_1 b + k_2 d)\boldsymbol{u}_2$$

- Thus the old coordinate vector for **v** is

$$[\boldsymbol{v}]_B = \begin{bmatrix} k_1 a + k_2 c \\ k_1 b + k_2 d \end{bmatrix}$$

which can be written as

$$[\boldsymbol{v}]_B = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \Rightarrow [\boldsymbol{v}]_B = \begin{bmatrix} a & c \\ b & d \end{bmatrix} [\boldsymbol{v}]_{B'}$$

- This equation states that the old coordinate vector results when we multiply the new coordinate on the left by the matrix

$$P = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

The columns are the coordinates of the new basis vectors relative to the old basis.

Solution of the Change of Basis Problem

- Solution of the change of basis problem
  - If we change the basis for a vector space V from some old basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ to some new basis $B' = \{\mathbf{u}_1', \mathbf{u}_2', \ldots, \mathbf{u}_n'\}$, then the old coordinate matrix $[\mathbf{v}]_B$ of a vector $\mathbf{v}$ is related to the new coordinate matrix $[\mathbf{v}]_{B'}$ of the same vector $\mathbf{v}$ by the equation

$$[\mathbf{v}]_B = P\,[\mathbf{v}]_{B'}$$

where the column of P are the coordinate matrices of the new basis vectors relative to the old basis; that is, the column vectors of P are

$$[\mathbf{u}_1']_B, [\mathbf{u}_2']_B, \ldots, [\mathbf{u}_n']_B$$

Transition Matrices

- Transition Matrices
  - The matrix P is called the transition matrix (轉移矩陣) from B' to B; it can be expressed in terms of its column vector as

$$P = [[\mathbf{u}_1']_B \mid [\mathbf{u}_2']_B \mid \ldots \mid [\mathbf{u}_n']_B]$$

Example (Finding a Transition Matrix)

- Consider bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}_1', \mathbf{u}_2'\}$ for R^2, where

  $$\mathbf{u}_1 = (1, 0),\ \mathbf{u}_2 = (0, 1);$$
  $$\mathbf{u}_1' = (1, 1),\ \mathbf{u}_2' = (2, 1).$$

  Find the transition matrix from B' to B.

  Find $[\mathbf{v}]_B$ if $[\mathbf{v}]_{B'} = [-3\ 5]^T$.

- Solution:
  - First we must find the coordinate matrices for the new basis vectors $\mathbf{u}_1'$ and $\mathbf{u}_2'$ relative to the old basis B.
  - By inspection $\mathbf{u}'_1 = \mathbf{u}_1 + \mathbf{u}_2$ so that

    $$\boldsymbol{u}_1' = \boldsymbol{u}_1 + \boldsymbol{u}_2$$
    $$\boldsymbol{u}_2' = 2\boldsymbol{u}_1 + \boldsymbol{u}_2$$

    $$[\mathbf{u}'_1]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } [\mathbf{u}'_2]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
  - Thus, the transition matrix from B' to B is $P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

Example (Finding a Transition Matrix)

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

- Using the transition matrix yields

$$[\boldsymbol{v}]_B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

- As a check, we should be able to recover the vector **v** either from $[\mathbf{v}]_B$ or $[\mathbf{v}]_{B'}$.
- $-3\mathbf{u}_1' + 5\mathbf{u}_2' = 7\mathbf{u}_1 + 2\mathbf{u}_2 = \mathbf{v} = (7,2)$

Example (A Different Viewpoint)

$$\mathbf{u}_1 = (1, 0),\ \mathbf{u}_2 = (0, 1);\ \mathbf{u}_1' = (1, 1),\ \mathbf{u}_2' = (2, 1)$$

- In the previous example, we found the transition matrix from the basis *B*' to the basis *B*. However, we can just as well ask for the transition matrix from *B* to *B*'.
- We simply change our point of view and regard *B*' as the old basis and *B* as the new basis.
- As usual, the columns of the transition matrix will be the coordinates of the new basis vectors relative to the old basis.

$$\mathbf{u}_1 = -\mathbf{u}_1' + \mathbf{u}_2'; \qquad \mathbf{u}_2 = 2\mathbf{u}_1' - \mathbf{u}_2'$$

$$[\boldsymbol{u}_1]_{B'} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad [\boldsymbol{u}_2]_{B'} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad Q = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

Remarks

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

- If we multiply the transition matrix from *B*' to *B* and the transition matrix from *B* to *B*', we find

$$PQ = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$Q = P^{-1}$$

Theorems

- Theorem 4.6.1
  - If P is the transition matrix from a basis B' to a basis B for a finite-dimensional vector space V, then:
    - P is invertible.
    - P^{-1} is the transition matrix from B to B'.

- Remark
  - If P is the transition matrix from a basis B' to a basis B, then for every **v** the following relationships hold:

$$[\mathbf{v}]_B = P\,[\mathbf{v}]_{B'}$$
$$[\mathbf{v}]_{B'} = P^{-1}\,[\mathbf{v}]_B$$

Computing Transition Matrices

- A procedure for computing $P_{B\to B'}$
- Step 1. Form the matrix [*B*'|*B*]
- Step 2. Use elementary row operations to reduce the matrix in Step 1 to reduced row echelon form
- Step 3. The resulting matrix will be $[\ I|P_{B\to B'}\ ]$
- Step 4. Extract the matrix $P_{B\to B'}$ from the right side of the matrix in Step 3.

Example

- Consider bases $B = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $B' = \{\mathbf{u}_1', \mathbf{u}_2'\}$ for R^2, where

$$\mathbf{u}_1 = (1, 0), \mathbf{u}_2 = (0, 1);$$
$$\mathbf{u}_1' = (1, 1), \mathbf{u}_2' = (2, 1).$$

- Find transition matrix from B' to B

$$\left[\text{old basis} \mid \text{new basis}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\Rightarrow \quad P_{B' \to B} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

- Find transition matrix from B to B'

$$\left[\text{new basis} \mid \text{old basis}\right] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

$$P_{B \to B'} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

Theorem 4.6.2

- Let $B'=\{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n\}$ be any basis for the vector space R^n and let $S=\{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n\}$ be the standard basis for R^n. If the vectors in these bases are written in column form, then

$$P_{B'\to S} = \left[\boldsymbol{u}_1|\boldsymbol{u}_2|\cdots|\boldsymbol{u}_n\right]$$

- If $A=[\mathbf{u}_1|\mathbf{u}_2|\ldots|\mathbf{u}_n]$ is any $n \times n$ invertible matrix, then A can be viewed as the transition matrix from the basis $\{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n\}$ for R^n to the standard basis for R^n.