Chapter 1
Systems of Linear Equations and Matrices

Outline

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1.1

Introduction to Systems of Linear Equations

Linear Equations

- Any straight line in *xy*-plane can be represented algebraically by an equation of the form:

$$a_1x + a_2y = b$$

- General form: Define a linear equation in the n variables $x_1, x_2, \ldots, x_n$:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ and b are real constants.

- The variables in a linear equation are sometimes called unknowns.

Example (Linear Equations)

- The equations $x+3y=7$, $y=\frac{1}{2}x+3z+1$, and $x_1-2x_2-3x_3+x_4=7$ are linear
  - A linear equation does not involve any products or roots of variables
  - All variables occur only to the first power and do not appear as arguments for trigonometric, logarithmic, or exponential functions.
- The equations $x+3\sqrt{y}=5$, $3x+2y-z+xz=4$, and $y=\sin x$ are *not* linear
- A solution of a linear equation is a sequence of *n* numbers s_1, s_2, …, s_n such that the equation is satisfied.
- The set of all solutions of the equation is called its solution set or general solution (通解) of the equation.

Example

- Find the solution of $x_1 - 4x_2 + 7x_3 = 5$
- Solution:
  - We can assign arbitrary values to any two variables and solve for the third variable
  - For example

$$x_1 = 5 + 4s - 7t, \qquad x_2 = s, \qquad x_3 = t$$

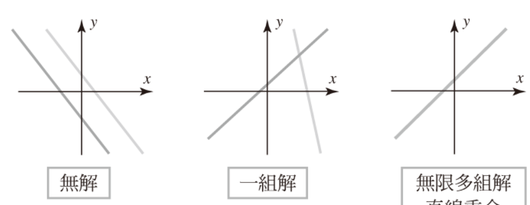
where s, t are arbitrary values

Linear Systems

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad \quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

- A finite set of linear equations in the variables $x_1, x_2, \ldots, x_n$ is called a system of linear equations or a linear system (線性系統).
- A sequence of numbers $s_1, s_2, \ldots, s_n$ is called a solution of the system
- A system has *no solution* is said to be inconsistent. (矛盾方程組)
- If there is at least one solution of the system, it is called consistent. (相容的)
- *Every system of linear equations has either no solutions, exactly one solution, or infinitely many solutions*
- A general system of two linear equations:

$$a_1x + b_1y = c_1 \ (a_1, b_1 \text{ not both zero})$$
$$a_2x + b_2y = c_2 \ (a_2, b_2 \text{ not both zero})$$

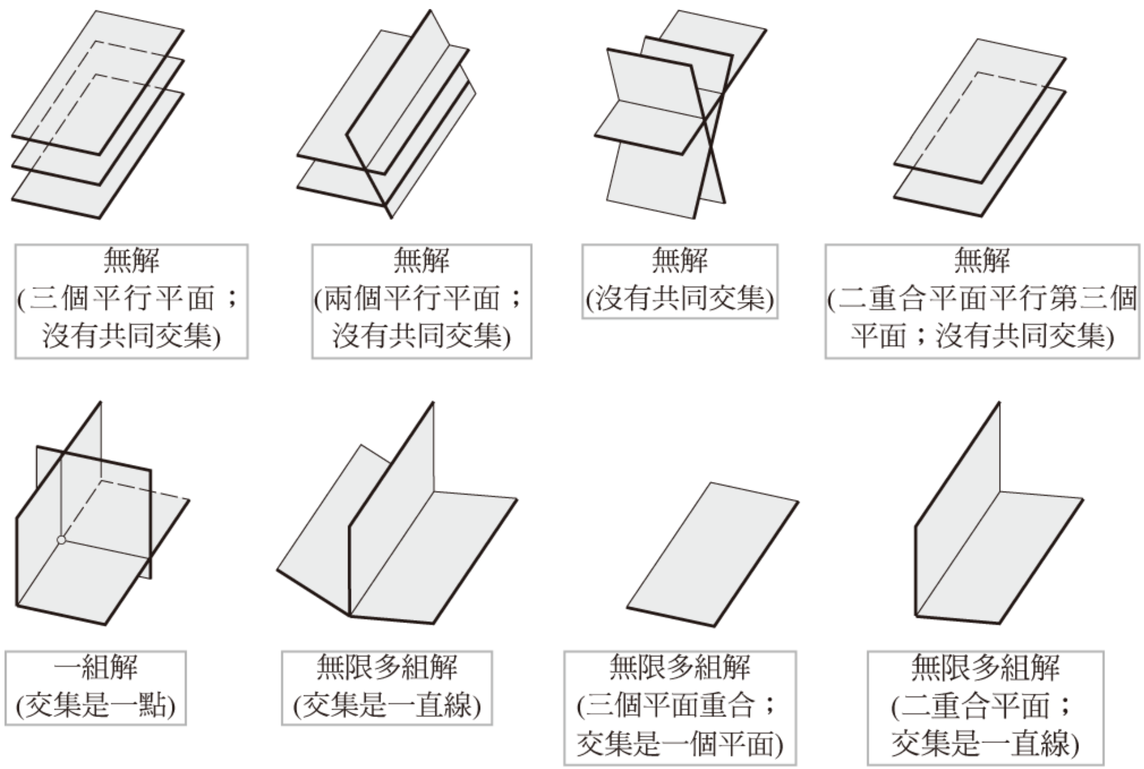


▲ 圖 1.1.1

  - Two lines may be parallel – no solution
  - Two lines may be intersect at only one point – one solution
  - Two lines may coincide – infinitely many solutions

Linear Systems

- Example of a linear system of three equations in three unknowns



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Example

$$\begin{aligned} x - y + 2z &= 5 \\ 2x - 2y + 4z &= 10 \\ 3x - 3y + 6z &= 15 \end{aligned}$$

After elimination ⇨ $x - y + 2z = 5$

General solution:

$$\begin{aligned} x &= 5 + r - 2s \\ y &= r \\ z &= s \end{aligned}$$

- The three planes coincide!

Augmented Matrices

- The location of the $+'$s, the x's, and the $='$s can be abbreviated by writing only the rectangular array of numbers.
- This is called the augmented matrix (增廣矩陣) for the system.
- It must be written in the same order in each equation as the unknowns and the constants must be on the right

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\quad & \vdots \quad\ \ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

1th column (欄) ↓

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

← 1th row (列)

Elementary Row Operations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

- The basic method for solving a system of linear equations is to replace the given system by **a new system that has the same solution set** but which is **easier** to solve.
- Since the **rows** of an augmented matrix correspond to the **equations** in the associated system, a new system is generally obtained in a series of steps by applying the following three types of operations to eliminate unknowns systematically.
- These are called elementary row operations
  - Multiply an equation by an nonzero constant
  - Interchange two equations
  - Add a multiple of one equation to another

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{bmatrix}$$

Example (Using Elementary Row Operations)

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned} \Rightarrow \begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3x + 6y - 5z &= 0 \end{aligned} \Rightarrow \begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned} \Rightarrow \begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ 3y - 11z &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ -\tfrac{1}{2}z &= -\tfrac{3}{2} \end{aligned} \Rightarrow \begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned} \Rightarrow \begin{aligned} x + \tfrac{11}{2}z &= \tfrac{35}{2} \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned} \Rightarrow \begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

1.2
Gaussian Elimination

Echelon Forms

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

- A matrix which has the following properties is in reduced row-echelon form (as in the previous example) (簡約列-梯型)
  - If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a leading 1. (首項1)
  - If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
  - In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
  - Each *column* that contains a leading 1 has zeros everywhere else.
- A matrix that has the *first three properties* is said to be in row-echelon form. (列-梯型)
- Note: A matrix in reduced row-echelon form is of necessity in row-echelon form, but not conversely.

Example

- Reduced row-echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- Row-echelon form:

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

- All matrices of the following types are in **row-echelon form** (any real numbers substituted for the *'s.) :

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

- All matrices of the following types are in **reduced row-echelon form** (any real numbers substituted for the *'s.) :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Example

- Suppose that the augmented matrix for a linear system in the unknowns *x*, *y*, *z* has been reduced as

$$\begin{bmatrix} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} x - 5y + z &= 4 \\ 0x + 0y + 0z &= 0 \end{aligned} \quad \Rightarrow \quad x = 4 + 5y - z$$

General solution:

$$\begin{aligned} x &= 4 + 5s - t \\ y &= s \\ z &= t \end{aligned}$$

s and *t* can be arbitrary values

Elimination Methods

- A step-by-step elimination procedure that can be used to reduce any matrix to reduced row-echelon form

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Elimination Methods

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

- Step1. Locate the leftmost column that does not consist entirely of zeros.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Leftmost nonzero column

- Step2. Interchange the top row with another row, to bring a nonzero entry to top of the column found in Step1

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

The 1th and 2th rows in the preceding matrix were interchanged.

Elimination Methods

- Step3. If the entry that is now at the top of the column found in Step1 is a, multiply the first row by 1/a in order to introduce a leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

The 1st row of the preceding matrix was multiplied by 1/2.

- Step4. Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

-2 times the 1st row of the preceding matrix was added to the 3rd row.

Elimination Methods

- Step5. Now cover the top row in the matrix and begin again with Step1 applied to the submatrix that remains. Continue in this way until the entire matrix is in row-echelon form

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

Leftmost nonzero column in the submatrix

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

The 1st row in the submatrix was multiplied by -1/2 to introduce a leading 1.

Elimination Methods

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

-5 times the 1st row of the submatrix was added to the 2nd row of the submatrix to introduce a zero below the leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

The top row in the submatrix was covered, and we returned again Step1.

Leftmost nonzero column in the new submatrix

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The first (and only) row in the new submetrix was multiplied by 2 to introduce a leading 1.

Elimination Methods

- Step 6. Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

← **7/2 times the third row was added to the second row**

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

← **-6 times the third row was added to the first row**

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

← **5 times the second row was added to the first row**

The **last** matrix is in **reduced row-echelon form**

Elimination Methods

- Step1~Step5: the above procedure produces a row-echelon form and is called Gaussian elimination (forward phase)
- Step1~Step6: the above procedure produces a reduced row-echelon form and is called Gaussian-Jordan elimination (forward +backward phases)
- Every matrix has **a unique reduced row-echelon** form but a row-echelon form of a given matrix is not unique
- Back-Substitution
  - It is sometimes preferable to solve a system of linear equations by using Gaussian elimination to bring the augmented matrix into row-echelon form **without continuing all the way to the reduced row-echelon form**.
  - When this is done, the corresponding system of equations can be solved by a technique called back-substitution

Homogeneous Linear Systems

- A system of linear equations is said to be homogeneous (齊次的) if the constant terms are all zero; that is, the system has the form:

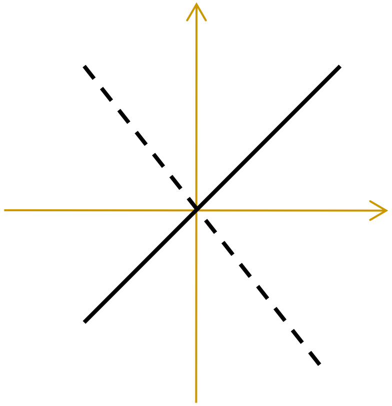
$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0
\end{aligned}
$$

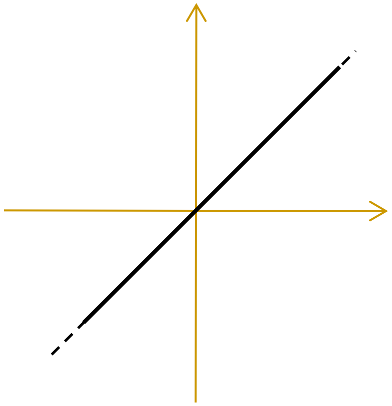
- Every homogeneous system of linear equation is **consistent**, since all such system have $x_1 = 0, x_2 = 0, \dots, x_n = 0$ as a solution.
  - This solution is called the trivial solution. (明顯解)
  - If there are another solutions, they are called nontrivial solutions.
- There are *only two possibilities* for its solutions:
  - There is **only** the trivial solution
  - There are **infinitely** many solutions in addition to the trivial solution

Example

- A homogeneous linear system of two equations in two unknowns

$$a_1 x + b_1 y = 0$$
$$a_2 x + b_2 y = 0$$





Example (Gauss-Jordan Elimination)

- Solve the homogeneous system of linear equations by Gauss-Jordan elimination

$$\begin{aligned} 2x_1 + 2x_2 - x_3 \quad\quad + x_5 &= 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 &= 0 \\ x_1 + x_2 - 2x_3 \quad\quad - x_5 &= 0 \\ x_3 + x_4 + x_5 &= 0 \end{aligned}$$

- The augmented matrix

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

- Reducing this matrix to reduced row-echelon form

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- The general solution is

$$x_1 = -s - t,\ x_2 = s$$
$$x_3 = -t,\ x_4 = 0,\ x_5 = t$$

- Note: the trivial solution is obtained when $s = t = 0$

Example (Gauss-Jordan Elimination)

- Two important points:
  - None of the three row operations alters the final column of zeros, so the system of equations corresponding to the reduced row-echelon form of the augmented matrix must also be a homogeneous system.
  - If the given homogeneous system has *m* equations in *n* unknowns with $m < n$, and there are *r* nonzero rows in reduced row-echelon form of the augmented matrix, we will have $r < n$. It will have the form:

$$\begin{array}{llll} \cdots x_{k1} & & + \sum () = 0 \\ & \cdots x_{k2} & + \sum () = 0 \\ & \ddots & \vdots \\ & & x_{kr} + \sum () = 0 \end{array} \qquad \begin{array}{l} x_{k1} = -\sum () \\ x_{k2} = -\sum () \\ \vdots \\ x_{kr} = -\sum () \end{array}$$

$$\begin{array}{ccccccc} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = & b_2 \\ \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = & b_m \end{array}$$

Theorem

- Theorem 1.2.1
  - If a homogeneous system has n unknowns, and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has n-r free variables.

- Theorem 1.2.2
  - A homogeneous system of linear equations with more unknowns than equations has **infinitely many solutions**.

Remarks

- Every matrix has a unique reduced row echelon form
- Row echelon forms are not unique
- Although row echelon forms are not unique, all row echelon forms of a matrix *A* have the same number of zero rows, and the leading 1's always occur in the same positions in the row echelon forms of *A*.

$$A = \begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

1.3

Matrices and Matrix Operations

Definition and Notation

- A matrix is a rectangular array of numbers. The numbers in the array are called the entries (元素) in the matrix
- A general $m \times n$ matrix A is denoted as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- The entry that occurs in row i and column j of matrix A will be denoted a_{ij} or $\langle A \rangle_{ij}$. If a_{ij} is real number, it is common to be referred as **scalars** (純量)
- The preceding matrix can be written as $[a_{ij}]_{m \times n}$ or $[a_{ij}]$
- A matrix A with n rows and n columns is called a square matrix of order n

Examples of Matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix} \qquad \begin{bmatrix} \pi & -\sqrt{2} \\ \frac{3}{2} & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 3 \end{bmatrix}$$

- A matrix with only one row is called a ***row matrix*** (or a ***row vector***).
- A matrix with only one column is called a ***column matrix*** (or a ***column vector***).

Sum, Difference, and Product

- Two matrices are defined to be equal if they have the same size and their corresponding entries are equal
  - If $A = [a_{ij}]$ and $B = [b_{ij}]$ have the same size, then $A = B$ if and only if $a_{ij} = b_{ij}$ for all i and j
- If A and B are matrices of the same size, then the sum $A + B$ is the matrix obtained by adding the entries of B to the corresponding entries of A.
- The difference $A - B$ is the matrix obtained by subtracting the entries of B from the corresponding entries of A
- If A is any matrix and c is any scalar, then the product cA is the matrix obtained by multiplying each entry of the matrix A by c. The matrix cA is said to be the scalar multiple of A
  - If $A = [a_{ij}]$, then $\langle cA \rangle_{ij} = c\langle A \rangle_{ij} = ca_{ij}$

Example

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 2 & 5 & 11 \\ 0 & 6 & -4 \end{bmatrix}$$

$$2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 2 & 1 & -3 \\ 2 & 0 & 6 \end{bmatrix}$$

linear combination:

$$2A + 3B = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 6 & 21 \\ -3 & 9 & -15 \end{bmatrix} = \begin{bmatrix} 4 & 12 & 29 \\ -1 & 15 & -13 \end{bmatrix}$$

Product of Matrices

- If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the product AB is the $m \times n$ matrix whose entries are determined as follows.
- To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and column together and then add up the resulting products
  - That is, $(AB)_{m \times n} = A_{m \times r} B_{r \times n}$

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix}$$

the entry $\langle AB \rangle_{ij}$ in row i and column j of AB is given by

$$\langle AB \rangle_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \ldots + a_{ir}b_{rj}$$

Product of Matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & 26 & \blacksquare \end{bmatrix}$$

$$2 \times 4 + 6 \times 3 + 0 \times 5 = 26$$

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \blacksquare & \blacksquare & \blacksquare & 13 \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{bmatrix}$$

$$1 \times 3 + 2 \times 1 + 4 \times 2 = 13$$

Example

- Determining whether a product is defined

  $A_{3\times 4}$ $B_{4\times 7}$ $C_{7\times 3}$

- *AB* is defined and is a 3×7 matrix; *BC* is defined and is a 4×3 matrix; and *CA* is defined and is a 7×4 matrix.
- The products *AC*, *CB*, and *BA* are all undefined.

Example

- If $A = [a_{ij}]$ is a $m \times r$ matrix and $B = [b_{ij}]$ is an $r \times n$ matrix, then the entry $(AB)_{ij}$ is given by

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix}$$

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj}$$

Partitioned Matrices

- A matrix can be subdivided or partitioned into smaller matrices by inserting horizontal and vertical rules between selected rows and columns
- For example, three possible partitions of a 3×4 matrix A:
  - The partition of A into four submatrices A_{11}, A_{12}, A_{21}, and A_{22}
  - The partition of A into its row matrices $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$
  - The partition of A into its column matrices $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$, and $\mathbf{c}_4$

$$A=\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A=\left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]=\begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}$$

$$A=\left[\begin{array}{c|c|c|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]=\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{c}_4 \end{bmatrix}$$

Multiplication by Columns and by Rows

- It is possible to compute a particular row or column of a matrix product AB without computing the entire product:

jth column matrix of AB = A[jth column matrix of B]

ith row matrix of AB = [ith row matrix of A]B

- If $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_m$ denote the row matrices of A and $\mathbf{b}_1$,$\mathbf{b}_2$, ...,$\mathbf{b}_n$ denote the column matrices of B,then

$$AB = A\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n\end{bmatrix} = \begin{bmatrix}A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_n\end{bmatrix}$$

$$AB = \begin{bmatrix}\mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m\end{bmatrix} B = \begin{bmatrix}\mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_m B\end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} \qquad AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 27 \\ -4 \end{bmatrix}$$

← Second column of *AB*

↑ A

↑ Second column of *B*

$$A\boldsymbol{b}_2$$

First row of *AB*

$$\begin{bmatrix} 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \end{bmatrix}$$

↑ First row of *A*

B

$$\boldsymbol{a}_1 B$$

Matrix Products as Linear Combinations

- Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- Then

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

- *The product A**x** of a matrix A with a column matrix **x** is a **linear combination** of the column matrices of A with the coefficients coming from the matrix **x***

Example

The matrix product

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

can be written as the linear combination of column matrices

$$2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

The matrix product

$$\begin{bmatrix} 1 & -9 & -3 \end{bmatrix} \begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -16 & -18 & 35 \end{bmatrix}$$

can be written as the linear combination of row matrices

$$1 \begin{bmatrix} -1 & 3 & 2 \end{bmatrix} - 9 \begin{bmatrix} 1 & 2 & -3 \end{bmatrix} - 3 \begin{bmatrix} 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -16 & -18 & 35 \end{bmatrix}$$

Example (Columns of a Product AB as Linear Combinations)

We showed in Example 5 that

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

The column matrices of AB can be expressed as linear combinations of the column matrices of A as follows:

$$\begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 27 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 7\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 30 \\ 26 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 5\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 13 \\ 12 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Matrix Form of a Linear System

- Consider any system of *m* linear equations in *n* unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad \Rightarrow \quad A\mathbf{x} = \mathbf{b}$$

- The matrix *A* is called the coefficient matrix of the system
- The augmented matrix of the system is given by

$$[A \mid \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

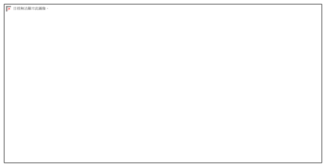
Matrices Defining Functions

- We can view *A* as defining a rule that shows how a given *x* is mapped into a corresponding *y*.

$$y = Ax = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -b \end{bmatrix}$$

- The effect of multiplying *A* by a column vector is to change the sign of the second entry of the column vector.

Matrices Defining Functions



$$y = Bx = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ -a \end{bmatrix}$$

- The effect of multiplying *B* by a column vector is to interchange the first and second entries of the column vector, also changing the sign of the first entry.

Transpose (轉置矩陣)

- If A is any $m \times n$ matrix, then the transpose of A, denoted by A^T, is defined to be the $n \times m$ matrix that results from interchanging the rows and columns of A
  - That is, the first column of A^T is the first row of A, the second column of A^T is the second row of A, and so forth
- If A is a square matrix, then the trace (跡數) of A , denoted by tr(A), is defined to be the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.
  - For an $n \times n$ matrix $A = [a_{ij}]$, 

Example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \quad D = \begin{bmatrix} 4 \end{bmatrix}$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix} \quad B^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix} \quad C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad D^T = \begin{bmatrix} 4 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$tr(A) = a_{11} + a_{22} + a_{33}$$

$$tr(B) = -1 + 5 + 7 + 0 = 11$$

1.4

Inverse; Algebraic Properties of Matrices

Properties of Matrix Operations

- For real numbers a and b ,we always have $ab = ba$, which is called the ***commutative law for multiplication***. For matrices, however, AB and BA need not be equal.
- Equality can fail to hold for three reasons:
  - The product AB is defined but BA is undefined.
  - AB and BA are both defined but have different sizes.
  - It is possible to have $AB \neq BA$ even if both AB and BA are defined and have the same size.

Example

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix} \qquad BA = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$$

$\Rightarrow$ $AB \neq BA$

Theorem 1.4.1 (Properties of Matrix Arithmetic)

- Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid:
  - $A + B = B + A$ (commutative law for addition)
  - $A + (B + C) = (A + B) + C$ (associative law for addition)
  - $A(BC) = (AB)C$ (associative law for multiplication)
  - $A(B + C) = AB + AC$ (left distributive law)
  - $(B + C)A = BA + CA$ (right distributive law)
  - $A(B - C) = AB - AC$, $(B - C)A = BA - CA$
  - $a(B + C) = aB + aC$, $a(B - C) = aB - aC$
  - $(a+b)C = aC + bC$, $(a-b)C = aC - bC$
  - $a(bC) = (ab)C$, $a(BC) = (aB)C = B(aC)$

Proof (d)

$A(B + C) = AB + AC$

- We must show that $A(B+C)$ and $AB+AC$ have the same size and that corresponding entries are equal.
- To form $A(B+C)$, the matrices B and C must have the same size, say $m \times n$, and the matrix A must then have m columns, so its size must be of the form $r \times m$. This makes $A(B+C)$ an $r \times n$ matrix.
- It follows that $AB+AC$ is also an $r \times n$ matrix.

Proof (d)

$A(B + C) = AB + AC$

- Suppose that $A = [a_{ij}]$, $B = [b_{ij}]$, and $C = [c_{ij}]$. We want to show

$$[A(B+C)]_{ij} = [AB + AC]_{ij}$$

- From the definitions of matrix addition and matrix multiplication, we have

$$\begin{aligned}
&[A(B+C)]_{ij} = a_{i1}(b_{1j}+c_{1j}) + a_{i2}(b_{2j}+c_{2j}) + \cdots + a_{im}(b_{mj}+c_{mj}) \\
&= (a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}) + (a_{i1}c_{1j} + a_{i2}c_{2j} + \cdots + a_{im}c_{mj}) \\
&= [AB]_{ij} + [AC]_{ij} \\
&= [AB + AC]_{ij}
\end{aligned}$$

Example

As an illustration of the associative law for matrix multiplication, consider

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad BC = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix}$$

Thus,

$$(AB)C = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$

and

$$A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$

so $(AB)C = A(BC)$, as guaranteed by Theorem 1.4.1*c*.

Zero Matrices (零矩陣)

- A matrix, all of whose entries are zero, is called a **zero matrix**
- A zero matrix will be denoted by 0
- If it is important to emphasize the size, we shall write $0_{m\times n}$ for the $m\times n$ zero matrix.
- In keeping with our convention of using **boldface symbols** for matrices with one column, we will denote a zero matrix with one column by $\mathbf{0}$
- Theorem 1.4.2 (Properties of Zero Matrices)
  - Assuming that the sizes of the matrices are such that the indicated operations can be performed ,the following rules of matrix arithmetic are valid
    - $A + 0 = 0 + A = A$
    - $A - A = 0$
    - $0 - A = -A$
    - $A0 = 0;\ 0A = 0$

Cancellation Law

- For real numbers:
  - If *ab*=*ac* and *a*≠0, then *b* = *c*
  - If *ab* = 0, then at least one of the factors on the left is 0.
- It fails in matrix operation

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$$

$$AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} \quad \text{but } B \neq C$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix} \qquad AB = \mathit{0} \text{ but } A \neq \mathit{0} \text{ and } B \neq \mathit{0}$$

Identity Matrices (單位矩陣)

- A square matrix with 1′s on the main diagonal and 0′s off the main diagonal is called an **identity matrix** and is denoted by I, or I_n for the $n \times n$ identity matrix
- If A is an $m \times n$ matrix, then $AI_n = A$ and $I_m A = A$
- An identity matrix plays the same role in matrix arithmetic as the number 1 plays in the numerical relationships $a \cdot 1 = 1 \cdot a = a$
- Theorem 1.4.3
  - If R is the reduced row-echelon form of an $n \times n$ matrix A, then either R has a row of zeros or R is the identity matrix I_n

Example

- Zero matrices

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 \end{bmatrix}$$

- Identity matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Proof of Theorem 1.4.3

- Suppose that the reduced row-echelon form of A is

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix}$$

- Either the last row in this matrix consists entirely of zeros or it does not.
- If not, the matrix contains no zero rows, and consequently each of the n rows has a leading entry of 1.

Proof of Theorem 1.4.3

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix}$$

- Since these leading 1's occur progressively father to the right as we move down the matrix, each of these 1's must occur on the main diagonal.
- Since the other entries in the same column as one of these 1's are zero, R must be I_n.
- Thus, either R has a row of zeros or $R = I_n$.

Inverse

- If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be *invertible* **(可逆的)** or *nonsingular* and B is called an *inverse* **(逆矩陣)** of A. If no such matrix B can be found, then A is said to be *singular*. (奇異的)

- Remark:
  - The inverse of A is denoted as A^{-1}
  - Not every (square) matrix has an inverse
  - An inverse matrix has exactly one inverse

Example

$$B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \quad \text{is an inverse of} \quad A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

A and *B* are invertible and each is an inverse of the other.

Example

The matrix $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$ is singular.

Let $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

The third column of *BA* is

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow BA \neq I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

*j*th column matrix of *BA* = *B*[*j*th column matrix of *A*]

Theorems

- Theorem 1.4.4
  - If B and C are both inverses of the matrix A, then $B = C$
- Theorem 1.4.5
  - The matrix
    $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
    is invertible if $ad - bc \neq 0$, in which case the inverse is given by the formula
    $$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
- Theorem 1.4.6
  - If A and B are invertible matrices of the same size ,then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

Proof of 1.4.4

If B and C are both inverses of the matrix A, then $B = C$

- Since B is an inverse of A, we have $BA = I$.
- Multiplying both sides on the right by C gives $(BA)C = IC = C$.
- But $(BA)C = B(AC) = BI = B$, so $C = B$.

Proof of 1.4.6

If A and B are invertible matrices of the same size ,then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

- If we can show that $(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I$, then we will have simultaneously shown that the matrix AB is invertible and that $(AB)^{-1} = B^{-1}A^{-1}$.
- $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$.
- A similar argument shows that $(B^{-1}A^{-1})(AB) = I$

Example

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \qquad AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix}$$

Applying the formula in Theorem 1.4.5, we obtain

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix} \quad (AB)^{-1} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

$$B^{-1}A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

Powers of A Matrix

- If A is a square matrix, then we define the nonnegative integer powers of A to be

$$A^0 = I \qquad A^n = \underbrace{AA\cdots A}_{n\text{ factors}} \quad (n > 0)$$

- If A is invertible, then we define the negative integer powers to be

$$A^{-n} = (A^{-1})^n = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{n\text{ factors}} \quad (n > 0)$$

Example

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

$$A^{-3} = (A^{-1})^3 = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

Theorems

- If A is a square matrix and r and s are integers, then $A^r A^s = A^{r+s}$, $(A^r)^s = A^{rs}$
- Theorem 1.4.7 (Laws of Exponents)
  - If A is invertible and n is a nonnegative integer, then:
    - A^{-1} is invertible and $(A^{-1})^{-1} = A$
    - A^n is invertible and $(A^n)^{-1} = (A^{-1})^n$ for $n = 0, 1, 2, \dots$
    - For any nonzero scalar k, the matrix kA is invertible and $(kA)^{-1} = k^{-1}A^{-1}=(1/k)A^{-1}$

Proof

A^{-1} is invertible and $(A^{-1})^{-1} = A$

- Since $AA^{-1} = A^{-1}A = I$, the matrix A^{-1} is invertible and $(A^{-1})^{-1} = A$.

For any nonzero scalar k, the matrix kA is invertible and $(kA)^{-1} = (1/k)A^{-1}$

$$(kA)(\frac{1}{k}A^{-1}) = \frac{1}{k}(kA)A^{-1} = (\frac{1}{k}k)AA^{-1} = 1I = I$$

Similarly, $(\frac{1}{k}A^{-1})(kA) = I$

Polynomial Expressions Involving Matrices

- If A is a square matrix, say $m \times m$, and if

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n$$

is any polynomial, then we define

$$p(A) = a_0 I + a_1 A + \ldots + a_n A^n$$

where I is the $m \times m$ identity matrix.

- That is, $p(A)$ is the $m \times m$ matrix that results when A is substituted for x in the above equation and a_0 is replaced by $a_0 I$

Example (Matrix Polynomial)

If

$$p(x) = 2x^2 - 3x + 4 \quad \text{and} \quad A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$$

then

$$p(A) = 2A^2 - 3A + 4I = 2\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 3\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 8 \\ 0 & 18 \end{bmatrix} - \begin{bmatrix} -3 & 6 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 0 & 13 \end{bmatrix}$$

Theorems

- Theorem 1.4.8 (Properties of the Transpose)
  - If the sizes of the matrices are such that the stated operations can be performed, then
    - $((A^T)^T = A$
    - $(A + B)^T = A^T + B^T$ and $(A – B)^T = A^T – B^T$
    - $(kA)^T = kA^T$, where k is any scalar
    - $(AB)^T = B^T A^T$
- Theorem 1.4.9 (Invertibility of a Transpose)
  - If A is an invertible matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$

Proof

If A is an invertible matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$

- We can prove the invertibility of A^T by showing that

$$A^T(A^{-1})^T = (A^{-1})^T A^T = I$$

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

$$A = \begin{bmatrix} -5 & -3 \\ 2 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} -5 & 2 \\ -3 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 3 \\ -2 & -5 \end{bmatrix} \quad (A^{-1})^T = \begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix} \qquad (A^T)^{-1} = \begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix}$$

1.5

Elementary Matrices and a Method for Finding A^{-1}

Elementary Row Operations

- An elementary row operation (sometimes called just a row operation) on a matrix A is any one of the following three types of operations:
  - Interchange of two rows of A
  - Replacement of a row $\mathbf{r}$ of A by $c\mathbf{r}$ for some number $c \neq 0$
  - Replacement of a row $\mathbf{r}_1$ of A by the sum $\mathbf{r}_1 + c\mathbf{r}_2$ of that row and a multiple of another row $\mathbf{r}_2$ of A
- Matrices A and B are ***row equivalent*** if either can be obtained from the other by a sequence of elementary row operations.

Elementary Matrices

- An $n \times n$ elementary matrix (基本矩陣) is a matrix produced by applying exactly one elementary row operation to I_n
  - E_{ij} is the elementary matrix obtained by interchanging the i-th and j-th rows of I_n
  - $E_i(c)$ is the elementary matrix obtained by multiplying the i-th row of I_n by $c \neq 0$
  - $E_{ij}(c)$ is the elementary matrix obtained by adding c times the j-th row to the i-th row of I_n, where $i \neq j$

Example (Elementary Matrices and Row Operations)

Listed below are four elementary matrices and the operations that produce them.

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$$

Multiply the second row of I_2 by -3.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Interchange the second and fourth rows of I_4.

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Add 3 times the third row of I_3 to the first row.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiply the first row of I_3 by 1.

Elementary Matrices and Row Operations

- Theorem 1.5.1 (Elementary Matrices and Row Operations)
  - Suppose that E is an $m \times m$ elementary matrix produced by applying a particular elementary row operation to I_m, and that A is an $m \times n$ matrix. Then EA is the matrix that results from applying that same elementary row operation to A
- Remark:
  - When a matrix A is multiplied on the **left** by an elementary matrix E, the effect is to perform an elementary row operation on A

Example (Using Elementary Matrices)

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

and consider the elementary matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

which results from adding 3 times the first row of I_3 to the third row. The product EA is

$$EA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}$$

which is precisely the same matrix that results when we add 3 times the first row of A to the third row. ◆

Inverse Operations

- If an elementary row operation is applied to an identity matrix I to produce an elementary matrix E, then there is a second row operation that, when applied to E, produces I back again

Example

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \xrightarrow{\text{Multiply the second row by 7}} \begin{bmatrix}1 & 0\\0 & 7\end{bmatrix} \xrightarrow{\text{Multiply the second row by 1/7}} \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \xrightarrow{\text{Interchange the first and the second rows}} \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \xrightarrow{\text{Interchange the first and the second rows}} \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \xrightarrow{\text{Add 5 times the second row to the first}} \begin{bmatrix}1 & 5\\0 & 1\end{bmatrix} \xrightarrow{\text{Add -5 times the second row to the first}} \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Theorem 1.5.2 (Elementary Matrices and Nonsingularity)

- Each elementary matrix is nonsingular (is invertible), and its inverse is itself an elementary matrix. More precisely,
  - $E_{ij}^{-1} = E_{ji}$ $(= E_{ij})$
  - $E_i(c)^{-1} = E_i(1/c)$ with $c \neq 0$
  - $E_{ij}(c)^{-1} = E_{ij}(-c)$ with $i \neq j$

Theorem 1.5.3 (Equivalent Statements)

- If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all true or all false
  - (a) A is invertible
  - (b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
  - (c) The reduced row-echelon form of A is I_n
  - (d) A is expressible as a product of elementary matrices

Proof

(a) $\rightarrow$ (b)

- Assume *A* is invertible and let $\boldsymbol{x}_0$ be any solution of $A\boldsymbol{x} = \mathbf{0}$ thus $A\boldsymbol{x}_0 = \mathbf{0}$.
- Multiplying both sides of this equation by the matrix A^{-1} gives $A^{-1}(A\boldsymbol{x}_0) = A^{-1} \cdot \mathbf{0}$, or $(A^{-1}A)\boldsymbol{x}_0 = \mathbf{0}$, or $I\boldsymbol{x}_0 = \mathbf{0}$, or $\boldsymbol{x}_0 = \mathbf{0}$.
- Thus, $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.

Proof

(b) → (c)

- Let $A\boldsymbol{x} = \mathbf{0}$ be the matrix form of the system

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & 0
\end{array}
$$

- Assume that the system has only the trivial solution. If we solve by Gauss-Jordan elimination, the reduced row-echelon form of the augmented matrix will be

$$
\begin{array}{cccccc}
x_1 & & & & = & 0 \\
& x_2 & & & = & 0 \\
& & \ddots & & & \\
& & & x_n & = & 0
\end{array}
$$

Proof

(b) → (c)

- The augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 \end{bmatrix}$$

can be reduced to the augmented matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

The reduced row-echelon form of A is I_n

Proof

(c) → (d)

- Assume that the reduced row-echelon form of A is I_n, so that A can be reduced to I_n by a finite sequence of elementary row operations.
- By Theorem 1.5.1, each of these operations can be accomplished by multiplying on the left by an appropriate elementary matrix. Thus we can find elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k \cdots E_2 E_1 A = I_n$$

Proof

$$E_k \cdots E_2 E_1 A = I_n$$

(c) → (d)

- By Theorem 1.5.2, E_1, E_2, …, E_k are invertible. Multiplying both sides on the left successively by $E_k^{-1}, \cdots, E_2^{-1}, E_1^{-1}$ we obtain

$$E_1^{-1} E_2^{-1} \cdots E_k^{-1} (E_k \cdots E_2 E_1 A) = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

- By Theorem 1.5.2, this equation expresses A as a product of elementary matrices.

A Method for Inverting Matrices

$$E_k \cdots E_2 E_1 A = I_n$$

- Multiplying on the right by A^{-1} yields

$$E_k \cdots E_2 E_1 A A^{-1} = I_n A^{-1}$$

$$A^{-1} = E_k \cdots E_2 E_1 I_n$$

- A^{-1} can be obtained by multiplying I_n successively on the left by the elementary matrices E_1, E_2, …, E_k .

- ***The sequence of row operations that reduces A to I_n will reduce I_n to A^{-1}.***

A Method for Inverting Matrices

- To find the inverse of an invertible matrix A, we must find a sequence of elementary row operations that reduces A to the identity and then perform this same sequence of operations on I_n to obtain A^{-1}

- Remark
  - Suppose we can find elementary matrices $E_1, E_2, \ldots, E_k$ such that

$$E_k \ldots E_2 E_1 A = I_n$$

then

$$A^{-1} = E_k \ldots E_2 E_1 I_n$$

Example (Using Row Operations to Find A^{-1})

- Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

- Solution:
  - To accomplish this we shall adjoin the identity matrix to the right side of A, thereby producing a matrix of the form $[A \mid I]$
  - We shall apply row operations to this matrix until the left side is reduced to I; these operations will convert the right side to A^{-1}, so that the final matrix will have the form $[I \mid A^{-1}]$

Example

The computations are as follows:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

← We added −2 times the first row to the second and −1 times the first row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

← We added 2 times the second row to the third.

Example

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

We multiplied the third row by -1.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

We added 3 times the third row to the second and -3 times the third row to the first.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

We added -2 times the second row to the first.

Thus,

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Example

- Not every matrix is invertible

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right]$$

We added -2 times the first row to the second and added the first row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

We added the second row to the third.

Since we have obtained a row of zeros on the left side, A is not invertible.

Example

- Determine whether the given homogeneous system has nontrivial solutions

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 + 5x_2 + 3x_3 &= 0 \\ x_1 + 8x_3 &= 0 \end{aligned} \qquad \begin{aligned} x_1 + 6x_2 + 4x_3 &= 0 \\ 2x_1 + 4x_2 - x_3 &= 0 \\ -x_1 + 2x_2 + 5x_3 &= 0 \end{aligned}$$

$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$ is invertible, and the first system has only trivial solution

$\begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$ is not invertible, and the second system has nontrivial solutions

1.6

More on Linear Systems and Invertible Matrices

Theorems

- Theorem 1.6.1
  - Every system of linear equations has either no solutions, exactly one solution, or in finitely many solutions.

- Theorem 1.6.2
  - If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\mathbf{b}$, the system of equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof of Theorem 1.6.1

- The proof will be complete if we can show that the system has infinitely many solutions if the system has more than one solution.
- Assume that $A\mathbf{x} = \mathbf{b}$ has more than one solution, and let $\mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_2$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are any two distinct solutions. Because $\mathbf{x}_1$ and $\mathbf{x}_2$ are distinct, $\mathbf{x}_0$ is nonzero.
- $A\mathbf{x}_0 = A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0}$
- $A(\mathbf{x}_1 + k\mathbf{x}_0) = A\mathbf{x}_1 + k(A\mathbf{x}_0) = \mathbf{b} + k\mathbf{0} = \mathbf{b} + \mathbf{0} = \mathbf{b}$. This says that $\mathbf{x}_1 + k\mathbf{x}_0$ is a solution of $A\mathbf{x} = \mathbf{b}$.
- Since $\mathbf{x}_0$ is nonzero and there are infinitely many choices for k, the system $A\mathbf{x}=\mathbf{b}$ has infinitely many solutions.

Example

Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 + 8x_3 &= 17 \end{aligned}$$

In matrix form this system can be written as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

In Example 4 of the preceding section we showed that A is invertible and

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

By Theorem 1.6.2 the solution of the system is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

or $x_1 = 1$, $x_2 = -1$, $x_3 = 2$.

Linear Systems with a Common Coefficient Matrix

- To solve a sequence of linear systems, $A\mathbf{x} = \mathbf{b}_1$, $A\mathbf{x} = \mathbf{b}_2$, …, $A\mathbf{x} = \mathbf{b}_k$, with common coefficient matrix A
- If A is invertible, then the solutions $\mathbf{x}_1 = A^{-1}\mathbf{b}_1$, $\mathbf{x}_2 = A^{-1}\mathbf{b}_2$, …, $\mathbf{x}_k = A^{-1}\mathbf{b}_k$
- A more efficient method is to form the matrix $[A|\mathbf{b}_1|\mathbf{b}_2|\ldots|\mathbf{b}_k]$
- By reducing it to reduced row-echelon form we can **solve all *k* systems at once** by Gauss-Jordan elimination.

Example

- Solve the systems

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_1 + 5x_2 + 3x_3 &= 5 \\ x_1 + 8x_3 &= 9 \end{aligned} \qquad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 5x_2 + 3x_3 &= 6 \\ x_1 + 8x_3 &= -6 \end{aligned}$$

$$\left[\begin{array}{ccc|c|c} 1 & 2 & 3 & 4 & 1 \\ 2 & 5 & 3 & 5 & 6 \\ 1 & 0 & 8 & 9 & -6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

$x_1 = 1, x_2 = 0, x_3 = 1$

$x_1 = 2, x_2 = 1, x_3 = -1$

Theorems

- Theorem 1.6.3
  - Let A be a square matrix
    - If B is a square matrix satisfying $BA = I$, then $B = A^{-1}$
    - If B is a square matrix satisfying $AB = I$, then $B = A^{-1}$

Proof of Theorem 1.6.3

- Assume that $BA=I$. If we can show that A is invertible, the proof can be completed by multiplying $BA = I$ on both sides by A^{-1} to obtain

$$BAA^{-1} = IA^{-1} \qquad BI = IA^{-1} \qquad B = A^{-1}$$

- To show that A is invertible, it suffices to show that the system $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.
- Let $\boldsymbol{x}_0$ be any solution of this system. If we multiply both sides of $A\boldsymbol{x}_0 = \mathbf{0}$ on the left by B, we obtain $BA\boldsymbol{x}_0 = B\mathbf{0}$ or $I\boldsymbol{x}_0 = \mathbf{0}$ or $\boldsymbol{x}_0 = \mathbf{0}$. Thus, the system $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.

Theorem 1.6.4 (Equivalent Statements)

- If A is an $n \times n$ matrix, then the following statements are equivalent
  - A is invertible
  - $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
  - The reduced row-echelon form of A is I_n
  - A is expressible as a product of elementary matrices
  - $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$
  - $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$

Theorems

- Theorem 1.6.5
  - Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.

- ***A fundamental problem***: Let A be a fixed $m \times n$ matrix. Find all $m \times 1$ matrices $\boldsymbol{b}$ such that the system of equations $A\mathbf{x} = \boldsymbol{b}$ is consistent.

Fundamental Problem

- If A is invertible, Theorem 1.6.2 says that $A\mathbf{x}=\mathbf{b}$ has the unique solution.
- If A is not square, or if A is square but not invertible
  - The matrix **b** must usually satisfy certain conditions in order for $A\mathbf{x}=\mathbf{b}$ to be consistent

Example

- What conditions must b_1, b_2, and b_3 satisfy in order for the systems of equations to be consistent?

$$x_1 + 2x_2 + 3x_3 = b_1$$
$$x_1 + x_3 = b_2$$
$$2x_1 + x_2 + 3x_3 = b_3$$

- The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{bmatrix}$$

Example

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{bmatrix}$$

-1 times the first row was added to the second and -2 times the first row was added to the third.

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & -1 & -1 & b_3 - 2b_1 \end{bmatrix}$$

The second row was multiplied by -1.

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}$$

The second row was multiplied to the third.

Example

$$\begin{bmatrix} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ 0 & 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}$$

- The system has a solution if and only if b_1, b_2, and b_3 satisfy the condition

$$b_3 - b_2 - b_1 = 0 \quad \text{or} \quad b_3 = b_2 + b_1$$

- To express this condition another way, $A\mathbf{x}=\mathbf{b}$ is consistent if and only if $\mathbf{b}$ is a matrix of the form

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_1 + b_2 \end{bmatrix}$$

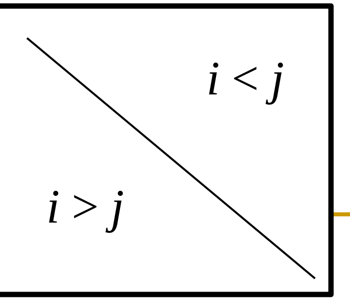
where b_1 and b_2 are arbitrary.

1.7

Diagonal, Triangular, and Symmetric Matrices

Diagonal and Triangular

- A square matrix A is $m \times n$ with $m = n$; the (i,i)-entries for $1 \le i \le m$ form the main diagonal of A
- A diagonal matrix (對角矩陣) is a square matrix all of whose entries *not* on the main diagonal equal zero. By diag($d_1, \ldots, d_m$) is meant the $m \times m$ diagonal matrix whose (i,i)-entry equals d_i for $1 \le i \le m$
- A $n \times n$ lower-triangular matrix (下三角矩陣) L satisfies $(L)_{ij} = 0$ if $i < j$, for $1 \le i \le m$ and $1 \le j \le n$
- A $n \times n$ upper-triangular matrix (上三角矩陣) U satisfies $(U)_{ij} = 0$ if $i > j$, for $1 \le i \le m$ and $1 \le j \le n$



Properties of Diagonal Matrices

- A general $n{\times}n$ diagonal matrix D can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

- A diagonal matrix is invertible if and only if all of its diagonal entries are nonzero

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix}$$

- Powers of diagonal matrices are easy to compute

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \qquad A^{5} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -243 & 0 \\ 0 & 0 & 32 \end{bmatrix} \qquad A^{-5} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{243} & 0 \\ 0 & 0 & \frac{1}{32} \end{bmatrix}$$

Properties of Diagonal Matrices

- Matrix products that involve diagonal factors are especially easy to compute

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_1a_{12} & d_1a_{13} & d_1a_{14} \\ d_2a_{21} & d_2a_{22} & d_2a_{23} & d_2a_{24} \\ d_3a_{31} & d_3a_{32} & d_3a_{33} & d_3a_{34} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_2a_{12} & d_3a_{13} \\ d_1a_{21} & d_2a_{22} & d_3a_{23} \\ d_1a_{31} & d_2a_{32} & d_3a_{33} \\ d_1a_{41} & d_2a_{42} & d_3a_{43} \end{bmatrix}$$

To multiply a matrix A on the left by a diagonal matrix D, one can multiply successive rows of A by the successive diagonal entries of D.
To multiply A on the right by D, one can multiply successive columns of A by the successive diagonal entries of D.

Theorem 1.7.1

- The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular
- The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular
- A triangular matrix is invertible if and only if its diagonal entries are all nonzero
- The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular

Example

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix A is invertible, since its diagonal entries are nonzero, but the matrix B is not.

$$A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

The product AB is also upper triangular.

$$AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

Proof

The product of lower triangular matrices is lower triangular.

- Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be lower triangular $n \times n$ matrices, and let $C = [c_{ij}]$ be the product *C*=*AB*.
- We can prove that *C* is lower triangular by showing that $c_{ij} = 0$ for *i* < *j*.

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

- If we assume that *i* < *j*, then the terms can be grouped as

$$c_{ij} = \underbrace{a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{i(j-1)}b_{(j-1)j}} + \underbrace{a_{ij}b_{jj} + \cdots + a_{in}b_{nj}}$$

The row number of *b* is less than the column number of *b*

The row number of *a* is less than the column number of *a*

Proof

The product of lower triangular matrices is lower triangular.

$$c_{ij} = \underbrace{a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{i(j-1)}b_{(j-1)j}}_{\text{The row number of } b \text{ is less than the column number of } b} + \underbrace{a_{ij}b_{jj} + \cdots + a_{in}b_{nj}}_{\text{The row number of } a \text{ is less than the column number of } a}$$

- In the first grouping all of the *b* factors are zero since *B* is lower triangular. In the second grouping all of the *a* factors are zero since *A* is lower triangular. Thus, $c_{ij}=0$.

Symmetric Matrices

- Definition
  - A (square) matrix A for which $A^T = A$, so that $\langle A \rangle_{ij} = \langle A \rangle_{ji}$ for all i and j, is said to be symmetric.
- Theorem 1.7.2
  - If A and B are symmetric matrices (對稱矩陣) with the same size, and if k is any scalar, then
    - A^T is symmetric
    - $A + B$ and $A - B$ are symmetric
    - kA is symmetric
- Theorem 1.7.3
  - The product of two symmetric matrices is symmetric if and only if the matrices commute (可交換), i.e., $AB = BA$

Example

- It is not true, in general, that the product of symmetric matrices is symmetric.

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}$$

- If these two matrices commute, the product of two symmetric matrices is symmetric.

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

Theorems

- Theorem 1.7.4
  - If A is an invertible symmetric matrix, then A^{-1} is symmetric.

- Remark:
  - In general, a symmetric matrix needs not be invertible.
  - The products AA^T and A^TA are always symmetric

- Theorem 1.7.5
  - If A is an invertible matrix, then AA^T and A^TA are also invertible

Proof

If A is an invertible symmetric matrix, then A^{-1} is symmetric.

- Assume that A is symmetric and invertible. From Theorem 1.4.9 and the fact that $A=A^T$, we have

$$(A^{-1})^T = (A^T)^{-1} = A^{-1}$$

which proves that A^{-1} is symmetric.

Theorem 1.4.9
$$(A^{-1})^T = (A^T)^{-1}$$

The products AA^T and A^TA are always symmetric

- $(AA^T)^T = (A^T)^T A^T = AA^T$
- $(A^TA)^T = A^T(A^T)^T = A^TA$

Example

Let A be the 2×3 matrix

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$$

Then

$$A^T A = \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -2 & -11 \\ -2 & 4 & -8 \\ -11 & -8 & 41 \end{bmatrix}$$

$$A A^T = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 21 & -17 \\ -17 & 34 \end{bmatrix}$$

Observe that $A^T A$ and $A A^T$ are symmetric as expected.

Proof

If A is an invertible matrix, then AA^T and A^TA are also invertible

- Since A is invertible, so is A^T by Theorem 1.4.9.
- Thus AA^T and A^TA are invertible, since they are the products of invertible matrices.

Theorem 1.4.9

$$(A^{-1})^T = (A^T)^{-1}$$