4.7
Row Space, Column Space, and Null Space

Row Space and Column Space

- Definition
  - If A is an $m \times n$ matrix, then the subspace of R^n spanned by the row vectors of A is called the row space (列空間) of A, and the subspace of R^m spanned by the column vectors is called the column space (行空間) of A.
  - The solution space of the homogeneous system of equation $A\mathbf{x} = \mathbf{0}$, which is a subspace of R^n, is called the null space (零核空間) of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Remarks

- In this section we will be concerned with **two questions**
  - What relationships exist between the solutions of a linear system $A\mathbf{x}=\mathbf{b}$ and the row space, column space, and null space of A.
  - What relationships exist among the row space, column space, and null space of a matrix.

Remarks

- It follows from Formula (10) of Section 1.3

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$A\boldsymbol{x} = x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \cdots + x_n\boldsymbol{c}_n = \boldsymbol{b}$$

- We conclude that ***A*x=b is consistent (相容的) if and only if b is expressible as a linear combination of the column vectors of *A* or, equivalently, if and only if b is in the column space of *A***.

Theorem 4.7.1

- Theorem 4.7.1
  - A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

Example

- Let $A\mathbf{x} = \mathbf{b}$ be the linear system $\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$

  Show that $\mathbf{b}$ is in the column space of A, and express $\mathbf{b}$ as a linear combination of the column vectors of A.

- Solution:
  - Solving the system by Gaussian elimination yields

    $$x_1 = 2, x_2 = -1, x_3 = 3$$

  - Since the system is consistent, $\mathbf{b}$ is in the column space of A.
  - Moreover, it follows that

    $$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

General and Particular Solutions

- Theorem 4.7.2
  - If $\mathbf{x}_0$ denotes any single solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$, and if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ form a basis for the null space of A, (that is, the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$), then every solution of $A\mathbf{x} = \mathbf{b}$ can be expressed in the form

$$\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

Conversely, for all choices of scalars $c_1, c_2, \ldots, c_k$, the vector $\mathbf{x}$ in this formula is a solution of $A\mathbf{x} = \mathbf{b}$.

Note that $\mathbf{x}_0$ is perpendicular to $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$

Refer also to **Theorem 3.4.4** on Page 152 of Textbook.
The general solution of a consistent linear system Ax=b can be obtained by adding any specific solution of Ax=b to the general solution of Ax=o.

Proof of Theorem 4.7.2

- Assume that $\mathbf{x}_0$ is any fixed solution of $A\mathbf{x}=\mathbf{b}$ and that $\mathbf{x}$ is an arbitrary solution. Then $A\mathbf{x}_0 = \mathbf{b}$ and $A\mathbf{x} = \mathbf{b}$.
- Subtracting these equations yields

$$A\mathbf{x} - A\mathbf{x}_0 = \mathbf{0} \quad \text{or} \quad A(\mathbf{x}\text{-}\mathbf{x}_0)=\mathbf{0}$$

- Which shows that $\mathbf{x}\text{-}\mathbf{x}_0$ is a solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$.
- Since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is a basis for the solution space of this system, we can express $\mathbf{x}\text{-}\mathbf{x}_0$ as a linear combination of these vectors, say $\mathbf{x}\text{-}\mathbf{x}_0 = c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_k\mathbf{v}_k$. Thus, $\mathbf{x}=\mathbf{x}_0+c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_k\mathbf{v}_k$.

Proof of Theorem 4.7.2

- Conversely, for all choices of the scalars $c_1, c_2, \ldots, c_k$, we have

$$A\mathbf{x} = A(\mathbf{x}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k)$$

$$A\mathbf{x} = A\mathbf{x}_0 + c_1(A\mathbf{v}_1) + c_2(A\mathbf{v}_2) + \ldots + c_k(A\mathbf{v}_k)$$

- But $\mathbf{x}_0$ is a solution of the nonhomogeneous system, and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are solutions of the homogeneous system, so the last equation implies that

$$A\mathbf{x} = \mathbf{b} + \mathbf{0} + \mathbf{0} + \ldots + \mathbf{0} = \mathbf{b}$$

- Which shows that $\mathbf{x}$ is a solution of $A\mathbf{x} = \mathbf{b}$.

Remark

- Remark
  - The vector $\mathbf{x}_0$ is called a particular solution (特解) of $A\mathbf{x} = \mathbf{b}$.
  - The expression $\mathbf{x}_0 + c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$ is called the general solution (通解) of $A\mathbf{x} = \mathbf{b}$, the expression $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$ is called the general solution of $A\mathbf{x} = \mathbf{0}$.
  - The general solution of $A\mathbf{x} = \mathbf{b}$ is the sum of any particular solution of $A\mathbf{x} = \mathbf{b}$ and the general solution of $A\mathbf{x} = \mathbf{0}$.

Example (General Solution of $A\mathbf{x} = \mathbf{b}$)

- The solution to the nonhomogeneous system

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \quad + 2x_5 \quad &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\ 5x_3 + 10x_4 \quad + 15x_6 &= 5 \\ 2x_1 + 5x_2 \quad + 8x_4 + 4x_5 + 18x_6 &= 6 \end{aligned}$$

is

$x_1 = -3r - 4s - 2t,\ x_2 = r,$
$x_3 = -2s,\ x_4 = s,$
$x_5 = t,\ x_6 = 1/3$

- The result can be written in vector form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3r-4s-2t \\ r \\ -2s \\ s \\ t \\ 1/3 \end{bmatrix} = \overbrace{\underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix}}_{\mathbf{x}_0}}^{\boldsymbol{x}_0} + \overbrace{\underbrace{r\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{x}}}^{\boldsymbol{x}}$$

which is the general solution.

- The vector $\mathbf{x}_0$ is a **particular solution** of nonhomogeneous system, and the linear combination $\mathbf{x}$ is the general solution of the homogeneous system.

Elementary Row Operation

- Performing an elementary row operation on an augmented matrix does not change the solution set of the corresponding linear system.
- It follows that applying an elementary row operation to a matrix A does not change the solution set of the corresponding linear system $A\mathbf{x}=\mathbf{0}$, or stated another way, it does not change the null space of A.

The solution space of the homogeneous system of equation $A\mathbf{x} = \mathbf{0}$, which is a subspace of R^n, is called the null space of A.

Example

- Find a basis for the nullspace of $A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$

- Solution
  - The nullspace of A is the solution space of the homogeneous system

$$\begin{aligned} 2x_1 + 2x_2 - x_3 \qquad\qquad + x_5 &= 0 \\ -x_1 - x_2 - 2x_3 - 3x_4 + x_5 &= 0 \\ x_1 + x_2 - 2x_3 \qquad\quad - x_5 &= 0 \\ x_3 + x_4 + x_5 &= 0 \end{aligned}$$

  - In Example 10 of Section 4.5 we showed that the vectors

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for the nullspace.

Theorems 4.7.3 and 4.7.4

- Theorem 4.7.3
  - Elementary row operations do not change the nullspace of a matrix.

- Theorem 4.7.4
  - Elementary row operations do not change the row space of a matrix.

Proof of Theorem 4.7.4

- Suppose that the row vectors of a matrix A are $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$, and let B be obtained from A by performing an elementary row operation. (We say that A and B are **row equivalent**.)
- We shall show that every vector in the row space of B is also in that of A, and that every vector in the row space of A is in that of B.
- If the row operation is a **row interchange**, then B and A have the same row vectors and consequently have the same row space.

Proof of Theorem 4.7.4

- If the row operation is **multiplication of a row by a nonzero scalar or a multiple of one row to another**, then the row vector $\mathbf{r}_1', \mathbf{r}_2', \ldots, \mathbf{r}_m'$ of B are linear combination of $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$; thus they lie in the row space of A.
- Since a vector space is closed under addition and scalar multiplication, all linear combination of $\mathbf{r}_1', \mathbf{r}_2', \ldots, \mathbf{r}_m'$ will also lie in the row space of A. Therefore, each vector in the row space of B is in the row space of A.

Proof of Theorem 4.7.4

- Since B is obtained from A by performing a row operation, A can be obtained from B by performing the **inverse operation** (Sec. 1.5).
- Thus the argument above shows that the row space of A is contained in the row space of B.

Remarks

- Do elementary row operations change the column space?
  - **Yes!**
- The second column is a scalar multiple of the first, so the column space of A consists of all scalar multiplies of the first column vector.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \xrightarrow[\text{row to the second}]{\text{Add -2 times the first}} B = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

non-parallel column vectors

parallel column vectors

- Again, the second column is a scalar multiple of the first, so the column space of B consists of all scalar multiples of the first column vector. This is not the same as the column space of A.

Theorem 4.7.5

- Theorem 4.7.5
  - If a matrix R is in row echelon form, then the row vectors with the leading 1's (i.e., the nonzero row vectors) form a basis for the row space of R, and the column vectors with the leading 1's of the row vectors form a basis for the column space of R.

(The proof involves little more than an analysis of the positions of the 0's and 1's of R. We omit the details.)

Bases for Row and Column Spaces

The matrix

$$R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

非leading 1's 的column 一定可以用它左邊是1's 的columns 的線性組合來表示(?)

is in row-echelon form. From Theorem 5.5.6 the vectors

$$\mathbf{r}_1 = [1\ \ -2\ \ 5\ \ 0\ \ 3]$$
$$\mathbf{r}_2 = [0\ \ 1\ \ 3\ \ 0\ \ 0]$$
$$\mathbf{r}_3 = [0\ \ 0\ \ 0\ \ 1\ \ 0]$$

form a basis for the row space of R, and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the column space of R.

Example

- Find bases for the row and column spaces of

$$A=\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

- Solution:
  - Since elementary row operations do not change the row space of a matrix, we can find a basis for the row space of A by finding a basis that of any row-echelon form of A.
  - Reducing A to row-echelon form we obtain

$$R=\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- The basis vectors for the row space of R and A

  $\mathbf{r}_1$ = [1 -3 4 -2 5 4]

  $\mathbf{r}_2$ = [0 0 1 3 -2 -6]

  $\mathbf{r}_3$ = [0 0 0 0 1 5]

- Keeping in mind that A and R may have different column spaces, **we cannot find a basis for the column space of *A* directly from the column vectors of *R***.

Theorem 4.7.6

- If A and B are row equivalent matrices, then:
  - A given set of column vectors of A is linearly independent if and only if the corresponding (對應的) column vectors of B are linearly independent.
  - A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.

(We omit the proofs here.)

Example

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- We can find the basis for the column space of R, then *the corresponding column vectors* of A will form a basis for the column space of A.
- Basis for R's column space

$$\boldsymbol{c}_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{c}_3' = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{c}_5' = \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

- Basis for A's column space

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \ \mathbf{c}_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \ \mathbf{c}_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

Example (Basis for a Vector Space Using Row Operations)

- Find a basis for the space spanned by the row vectors

$$\mathbf{v}_1 = (1, -2, 0, 0, 3),\ \mathbf{v}_2 = (2, -5, -3, -2, 6),$$
$$\mathbf{v}_3 = (0, 5, 15, 10, 0),\ \mathbf{v}_4 = (2, 6, 18, 8, 6).$$

- Except for a variation in notation, the space spanned by these vectors is the row space of the matrix

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

  - The nonzero row vectors in this matrix are

$$\mathbf{w}_1 = (1, -2, 0, 0, 3),\ \mathbf{w}_2 = (0, 1, 3, 2, 0),\ \mathbf{w}_3 = (0, 0, 1, 1, 0)$$

  - These vectors form a basis for the row space and consequently form a basis for the subspace of R^5 spanned by $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$.

Remarks

- Keeping in mind that A and R may have different column spaces, we cannot find a basis for the column space of A *directly* from the column vectors of R.
- However, if we can find a set of column vectors of R that forms a basis for the column space of R, then the *corresponding* column vectors of A will form a basis for the column space of A.

- **The basis vectors obtained for the column space of *A* consist of column vectors of *A*, but the basis vectors obtained (through a series of row operations) for the row space of *A* were not all vectors of *A*.**
- **Transpose of the matrix can be used to solve this problem**.

Example (Basis for the Row Space of a Matrix)

- Find a basis for the row space of

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

<u>consisting entirely of row vectors from *A*</u>.

- Solution:

$$A^T = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -2 & -5 & 5 & 6 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 3 & 6 & 0 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 5 & -10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- The column space of A^T are

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \ \mathbf{c}_2 = \begin{bmatrix} 2 \\ -5 \\ -3 \\ -2 \\ 6 \end{bmatrix}, \text{ and } \mathbf{c}_4 = \begin{bmatrix} 2 \\ 6 \\ 18 \\ 8 \\ 6 \end{bmatrix}$$

- Thus, the basis vectors for the row space of *A* are

$\mathbf{r}_1 = [1\ -2\ 0\ 0\ 3]$

$\mathbf{r}_2 = [2\ -5\ -3\ -2\ 6]$

$\mathbf{r}_3 = [2\ 6\ 18\ 8\ 6]$

Example (Basis and Linear Combinations)

- (a) Find a subset of the vectors $\mathbf{v}_1 = (1, -2, 0, 3)$, $\mathbf{v}_2 = (2, -5, -3, 6)$, $\mathbf{v}_3 = (0, 1, 3, 0)$, $\mathbf{v}_4 = (2, -1, 4, -7)$, $\mathbf{v}_5 = (5, -8, 1, 2)$ that forms a basis for the space spanned by these vectors.
- (b) Express each vector not in the basis as a linear combination of the basis vectors.

- Solution (a):

$$\begin{array}{ccccc} \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix} \\ \uparrow\ \ \uparrow\ \ \uparrow\ \ \uparrow\ \ \uparrow \\ \mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3\ \ \mathbf{v}_4\ \ \mathbf{v}_5 \end{array} \Rightarrow \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \uparrow\ \ \uparrow\ \ \uparrow\ \ \uparrow\ \ \uparrow \\ \mathbf{w}_1\ \ \mathbf{w}_2\ \ \mathbf{w}_3\ \ \mathbf{w}_4\ \ \mathbf{w}_5 \end{array}$$

  - Thus, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a basis for the column space of the matrix.

Example

- Solution (b):
  - We can express $\mathbf{w}_3$ as a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$, express $\mathbf{w}_5$ as a linear combination of $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_4$ (Why?). By inspection, these linear combination are

$$\mathbf{w}_3 = 2\mathbf{w}_1 - \mathbf{w}_2$$
$$\mathbf{w}_5 = \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_4$$

  - We call these the dependency equations. The corresponding relationships in the original vectors are

$$\mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2$$
$$\mathbf{v}_5 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_4$$

4.8
Rank, Nullity, and the Fundamental Matrix Spaces

Dimension and Rank

- Theorem 4.8.1
  - If A is any matrix, then the row space and column space of A have the same dimension.
- Proof: Let R be any row-echelon form of A. It follows from Theorem 4.7.4 and 4.7.6b that

$$\dim(\text{row space of } A) = \dim(\text{row space of } R).$$

$$\dim(\text{column space of } A) = \dim(\text{column space of } R)$$

- The dimension of the row space of R is the number of nonzero rows = number of leading 1's = dimension of the column space of R

Rank and Nullity

- Definition
  - The common dimension of the row and column space of a matrix A is called the rank (秩) of A and is denoted by rank(A); the dimension of the nullspace of a is called the nullity (零核維數) of A and is denoted by nullity(A).

Example (Rank and Nullity)

- Find the rank and nullity of the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

- Solution:
  - The reduced row-echelon form of A is

$$\begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

  - Since there are two nonzero rows (two leading 1's), the row space and column space are both two-dimensional, so rank(A) = 2.

Example (Rank and Nullity)

- To find the nullity of A, we must find the dimension of the solution space of the linear system $A\mathbf{x}=\mathbf{0}$.
- The corresponding system of equations will be

$$x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 = 0$$
$$x_2 - 2x_3 - 12x_4 - 16\,x_5 + 5\,x_6 = 0$$

- It follows that the general solution of the system is

$$x_1 = 4r + 28s + 37t - 13u,\ x_2 = 2r + 12s + 16t - 5u,$$
$$x_3 = r,\ x_4 = s,\ x_5 = t,\ x_6 = u$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus, nullity(A) = 4.

Example

- What is the maximum possible rank of an $m \times n$ matrix A that is not square?
- Solution: The row space of A is at most n-dimensional and the column space is at most m-dimensional. Since the rank of A is the common dimension of its row and column space, it follows that the rank is at most the smaller of m and n.

$$rank(A) \leq \min(m, n)$$

Theorem 4.8.2

- Theorem 4.8.2 (Dimension Theorem for Matrices)
  - If A is a matrix with n columns, then $\text{rank}(A) + \text{nullity}(A) = n$.
- Proof:
- Since A has n columns, $A\mathbf{x} = \mathbf{0}$ has n unknowns. These fall into two categories: the leading variables and the free variables.

$$\begin{bmatrix} \text{number of} \\ \text{leading variables} \end{bmatrix} + \begin{bmatrix} \text{number of} \\ \text{free variables} \end{bmatrix} = n$$

- The number of leading 1's in the reduced row-echelon form of A is the rank of A

$$\text{rank}(A) + \begin{bmatrix} \text{number of} \\ \text{free variables} \end{bmatrix} = n$$

Theorem 4.8.2

- The number of free variables is equal to the nullity of A. This is so because the nullity of A is the dimension of the solution space of $A\mathbf{x}=\mathbf{0}$, which is the same as the number of parameters in the general solution, which is the same as the number of free variables. Thus

$$\mathrm{rank}(A) + \mathrm{nullity}(A) = n$$

Example

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

- This matrix has 6 columns, so rank(A) + nullity(A) = 6
- In previous example, we know rank(A) = 4 and nullity(A) = 2

Theorem 4.8.3

- If A is an $m \times n$ matrix, then:
  - rank(A) = Number of leading variables in the solution of $A\mathbf{x} = \mathbf{0}$.
  - nullity(A) = Number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$.

$$x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 = 0$$
$$x_2 - 2x_3 - 12x_4 - 16\,x_5 + 5\,x_6 = 0$$

$$x_1 = 4r + 28s + 37t - 13u,\ x_2 = 2r + 12s + 16t - 5u,$$
$$x_3 = r,\ x_4 = s,\ x_5 = t,\ x_6 = u$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Example

- Find the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$ if A is a 5×7 matrix of rank 3.
- Solution:
  - $\text{nullity}(A) = n - \text{rank}(A) = 7 - 3 = 4$
  - Thus, there are four parameters.

Theorem 4.8.4 (Equivalent Statements)

- If A is an $n \times n$ matrix, and if $T_A : R^n \to R^n$ is multiplication by A, then the following are equivalent:
  - A is invertible.
  - $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  - The reduced row-echelon form of A is I_n.
  - A is expressible as a product of elementary matrices.
  - $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
  - $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
  - $\det(A) \neq 0$.
  - The column vectors of A are linearly independent.
  - The row vectors of A are linearly independent.
  - The column vectors of A span R^n.
  - The row vectors of A span R^n.
  - The column vectors of A form a basis for R^n.
  - The row vectors of A form a basis for R^n.
  - A has rank n.
  - A has nullity 0.

Overdetermined System

- A linear system with more equations than unknowns is called an overdetermined linear system (超定線性方程組). With fewer unknowns than equations, it's called an underdetermined linear system (欠定線性方程組).
- Theorem 4.8.5
  - If $A\mathbf{x} = \mathbf{b}$ is a consistent linear system of m equations in n unknowns, and if A has rank r, then the general solution of the system contains $n - r$ parameters.

- If A is a 5×7 matrix with rank 4, and if $A\mathbf{x}=\mathbf{b}$ is a consistent linear system, then the general solution of the system contains 7-4=3 parameters.

Theorem 4.8.6

- Let A be an $m \times n$ matrix
- (a) (Overdetemined Case) If $m > n$, then the linear system $A\mathbf{x}=\mathbf{b}$ is inconsistent for at least one vector $\mathbf{b}$ in R^m.
- (b) (Underdetermined Case) If $m < n$, then for each vector $\mathbf{b}$ in R^m the linear system $A\mathbf{x}=\mathbf{b}$ is either inconsistent or has infinitely many solutions.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Proof of Theorem 4.8.6 (a)

- Assume that $m>n$, in which case the column vectors of A cannot span R^m (fewer vectors than the dimension of R^m). Thus, there is at least one vector **b** in R^m that is not in the column space of A, and for that **b** the system $A\mathbf{x}=\mathbf{b}$ is inconsistent by Theorem 4.7.1.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Proof of Theorem 4.8.6 (b)

- Assume that $m<n$. For each vector **b** in R^n there are two possibilities: either the system $A\mathbf{x}=\mathbf{b}$ is consistent or it is inconsistent.
- If it is inconsistent, then the proof is complete.
- If it is consistent, then Theorem 4.8.5 implies that the general solution has n-r parameters, where r=rank(A).
- But rank(A) is smaller than, or equal to, the smaller of m and n, so $n-r \geqq n-m > 0$
- This means that the general solution has at least one parameter and hence there are infinitely many solutions.

Example

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- What can you say about the solutions of an **overdetermined** system $A\mathbf{x}=\mathbf{b}$ of 7 equations in 5 unknowns in which A has rank = 4?
- What can you say about the solutions of an **underdetermined** system $A\mathbf{x}=\mathbf{b}$ of 5 equations in 7 unknowns in which A has rank = 4?
- Solution:
  - (a) the system is consistent for some vector **b** in R^7, and for any such **b** the number of parameters in the general solution is $n-r=5-4=1$ (consistent 可能性會較低)
  - (b) the system may be consistent or inconsistent, but if it is consistent for the vector **b** in R^5, then the general solution has $n-r=7-4=3$ parameters. (consistent 可能性會較高)

Example

- The linear system
$$\begin{aligned} x_1 - 2x_2 &= b_1 \\ x_1 - x_2 &= b_2 \\ x_1 + x_2 &= b_3 \\ x_1 + 2x_2 &= b_4 \\ x_1 + 3x_2 &= b_5 \end{aligned}$$
is overdetermined, so it cannot be consistent for all possible values of b_1, b_2, b_3, b_4, and b_5. Exact conditions under which the system is consistent can be obtained by solving the linear system by Gauss-Jordan elimination.

$$\begin{bmatrix} 1 & 0 & 2b_2 - b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 - 3b_2 + 2b_1 \\ 0 & 0 & b_4 - 4b_2 + 3b_1 \\ 0 & 0 & b_5 - 5b_2 + 4b_1 \end{bmatrix}$$

Example

- Thus, the system is consistent if and only if b_1, b_2, b_3, b_4, and b_5 satisfy the conditions

$$\begin{aligned} 2b_1 - 3b_2 + b_3 \qquad\qquad &= 0 \\ 2b_1 - 4b_2 \qquad + b_4 \qquad &= 0 \\ 4b_1 - 5b_2 \qquad\qquad + b_5 &= 0 \end{aligned}$$

or, on solving this homogeneous linear system, $b_1=5r-4s$, $b_2=4r-3s$, $b_3=2r-s$, $b_4=r$, $b_5=s$ where r and s are arbitrary.

Fundamental Spaces of a Matrix

- Six important vector spaces associated with a matrix A
- **Row space of *A***, row space of A^T
- **Column space of *A***, column space of A^T
- **Null space of *A*, null space of A^T**
- Transposing a matrix converts row vectors into column vectors
  - Row space of A^T = column space of A
  - Column space of A^T = row space of A
- These are called the fundamental spaces of a matrix A

Theorem 4.8.7

- if A is any matrix, then rank(A) = rank(A^T)
- Proof:
  - Rank(A) = dim(row space of A) = dim(column space of A^T) = rank(A^T)
- If A is an $m \times n$ matrix, then rank(A)+nullity(A)=n. rank(A^T)+nullity(A^T) = m
- The dimensions of fundamental spaces

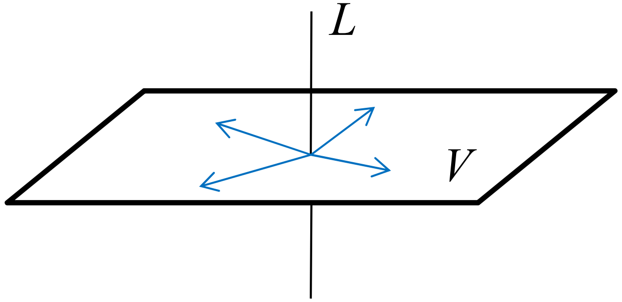
Fundamental Space	Dimension
Row space of A	r
Column space of A	r
Nullspace of A	$n - r$
Nullspace of A^T	$m - r$

Recap

- Theorem 3.4.3: If A is an $m \times n$ matrix, then the solution set of the homogeneous linear system $A\mathbf{x}=\mathbf{0}$ consists of all vectors in R^n that are orthogonal to every row vector of A.
- In other words, the null space of A consists of those vectors that are orthogonal to each of the row vectors of A.

Orthogonality

- Definition
  - Let W be a subspace of R^n, the set of all vectors in R^n that are orthogonal to every vector in W is called the orthogonal complement (正交補餘) of W, and is denoted by $W^{\perp}$
  - If V is a plane through the origin of R^3 with Euclidean inner product, then **the set of all vectors that are orthogonal to every vector in V forms the line L through the origin** that is perpendicular to V.



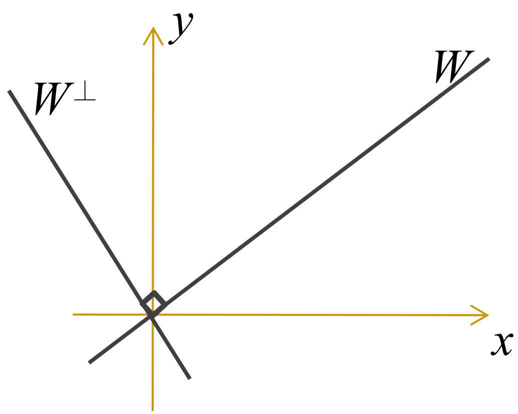
Theorem 4.8.8

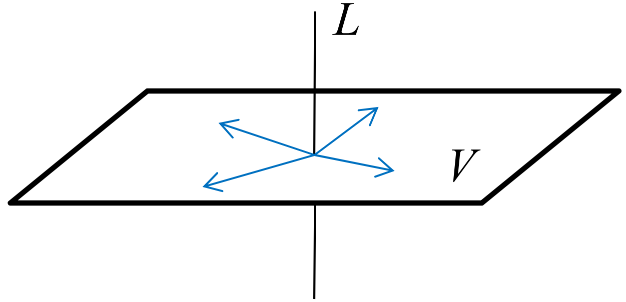
- Theorem 4.8.8
- If W is a subspace of a finite-dimensional space R^n, then:
  - $W^\perp$ is a subspace of R^n. (read "W perp")
  - The only vector common to W and $W^\perp$ is $\mathbf{0}$; that is ,$W \cap W^\perp = \mathbf{0}$.
  - The orthogonal complement of $W^\perp$ is W; that is , $(W^\perp)^\perp = W$.

$W \cup W^\perp = R^n$ (???)

Example

- Orthogonal complements

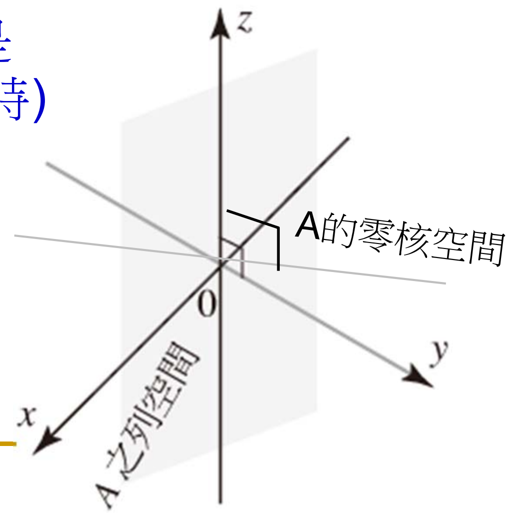


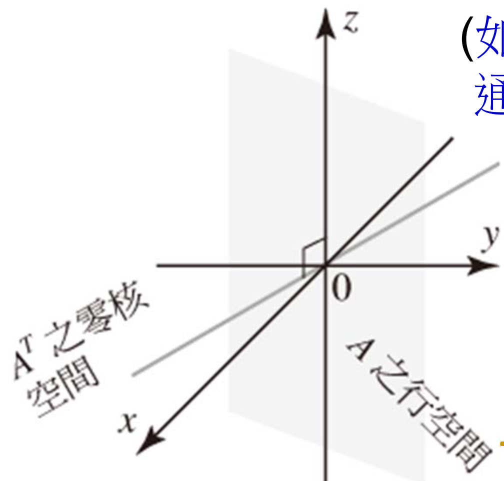


Theorem 4.8.9

- Theorem 4.8.9
  - If A is an $m \times n$ matrix, then:
    - The null space of A and the row space of A are orthogonal complements in R^n.
    - The null space of A^T and the column space of A are orthogonal complements in R^m.

(如果row space是通過原點的平面時)





(如果column space是通過原點的平面時)

Theorem 4.8.10 (Equivalent Statements)

- If A is an $m \times n$ matrix, and if $T_A : R^n \to R^n$ is multiplication by A, then the following are equivalent:
  - A is invertible.
  - $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  - The reduced row-echelon form of A is I_n.
  - A is expressible as a product of elementary matrices.
  - $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
  - $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
  - $\det(A) \neq 0$.
  - The column vectors of A are linearly independent.
  - The row vectors of A are linearly independent.
  - The column vectors of A span R^n.
  - The row vectors of A span R^n.
  - The column vectors of A form a basis for R^n.
  - The row vectors of A form a basis for R^n.
  - A has rank n.
  - A has nullity 0.
  - The orthogonal complement of the nullspace of A is R^n.
  - The orthogonal complement of the row space of A is $\{\mathbf{0}\}$.

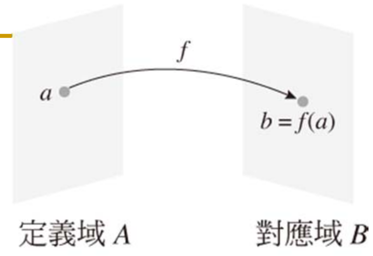
Applications of Rank

- Digital data are commonly stored in matrix form.
- **Rank plays a role because it measures the "redundancy" in a matrix.**
- If A is an $m \times n$ matrix of rank k, then n-k of the column vectors and m-k of the row vectors can be expressed in terms of k linearly independently column or row vectors.
- The essential idea in many data compression schemes is to approximate the original data set by a data set with smaller rank that conveys nearly the same information.

A => A'=$U\Sigma V^T$ (for example, SVD), where A' has a lower rank than A

4.9
Matrix Transformations from R^n to R^m

Functions from R^n to R



domain codomain

- A function is a rule f that associates with each element in a set A one and only one element in a set B.
- If f associates the element a with the element b, then we write $b = f(a)$ and say that b is the image of a under f or that $f(a)$ is the value of f at a.
- The set A is called the domain (定義域) of f and the set B is called the codomain (對應域) of f.
- The subset of the codomain B consisting of all possible values for f as a varies over A is called the **range** (值域) of f.

Examples

Formula	Example	Classification	Description
$f(x)$	$f(x) = x^2$	Real-valued function of a real variable	Function from R to R
$f(x, y)$	$f(x, y) = x^2 + y^2$	Real-valued function of two real variables	Function from R^2 to R
$f(x, y, z)$	$f(x, y, z) = x^2 + y^2 + z^2$	Real-valued function of three real variables	Function from R^3 to R
$f(x_1, x_2, \ldots, x_n)$	$f(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \ldots + x_n^2$	Real-valued function of n real variables	Function from R^n to R

Function from R^n to R^m

- Suppose $f_1, f_2, \ldots, f_m$ are real-valued functions of n real variables, say

$$w_1 = f_1(x_1, x_2, \ldots, x_n)$$
$$w_2 = f_2(x_1, x_2, \ldots, x_n)$$
$$\cdots$$
$$w_m = f_m(x_1, x_2, \ldots, x_n)$$

These m equations assign a unique point $(w_1, w_2, \ldots, w_m)$ in R^m to each point $(x_1, x_2, \ldots, x_n)$ in R^n and thus define a transformation from R^n to R^m. If we denote this transformation by T: $R^n \to R^m$ then

$$T\,(x_1, x_2, \ldots, x_n) = (w_1, w_2, \ldots, w_m)$$

Function from R^n to R^m

- If $m = n$ the transformation T: $R^n \rightarrow R^m$ is called an **operator** (運算子) on R^n.

Example: A Transformation from R^2 to R^3

$$w_1 = x_1 + x_2$$
$$w_2 = 3x_1x_2$$
$$w_3 = x_1^2 - x_2^2$$

- Define a (non-linear) transform T: $R^2 \rightarrow R^3$

(the motivation usually is to project lower-dimensional data points into a higher-dimensional space for better discrimination)

- With this transformation, the image of the point (x_1, x_2) is

$$T(x_1, x_2) = (x_1 + x_2, 3x_1x_2, x_1^2 - x_2^2)$$

- Thus, for example, $T(1,-2) = (-1, -6, -3)$

Linear Transformations from R^n to R^m

- A linear transformation (or a linear operator if $m = n$) $T: R^n \to R^m$ is defined by equations of the form

$$\begin{aligned} w_1 &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ w_2 &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \quad & \quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ w_m &= a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{aligned}$$

or

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{13} \\ a_{21} & a_{22} & \cdots & a_{23} \\ \vdots & \vdots & & \vdots \\ a_{mn} & a_{mn} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

or

$$\mathbf{w} = A\mathbf{x}$$

- The matrix $A = [a_{ij}]$ is called the standard matrix for the linear transformation T, and T is called multiplication by A.

Example (Transformation and Linear Transformation)

- The linear transformation $T : R^4 \rightarrow R^3$ defined by the equations

$$w_1 = 2x_1 - 3x_2 + x_3 - 5x_4$$
$$w_2 = 4x_1 + x_2 - 2x_3 + x_4$$
$$w_3 = 5x_1 - x_2 + 4x_3$$

the standard matrix for T (i.e., $\mathbf{w} = A\mathbf{x}$) is

$$A = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Notations

- If it is important to emphasize that A is the standard matrix for T, we denote the linear transformation T: $R^n \rightarrow R^m$ by T_A: $R^n \rightarrow R^m$. Thus,

$$T_A(\text{x}) = A\mathbf{x}$$

- We can also denote **the standard matrix** for T by the symbol $[T]$, or

$$T(\mathbf{x}) = [T]\mathbf{x}$$

Theorem 4.9.1

- For every matrix A the matrix (linear) transformation $T_A: R^n \rightarrow R^m$ has the following properties for all vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n and for every scalar k
  - (a) $T_A(\mathbf{0}) = \mathbf{0}$
  - (b) $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$ [Homogeneity property]
  - $T_A(\mathbf{u}+\mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$ [Additivity property]
  - $T_A(\mathbf{u}-\mathbf{v}) = T_A(\mathbf{u}) - T_A(\mathbf{v})$
- Proof: $A\mathbf{0} = \mathbf{0}$, $A(k\mathbf{u}) = k(A\mathbf{u})$, $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$,
  $A(\mathbf{u}-\mathbf{v})=A\mathbf{u}-A\mathbf{v}$

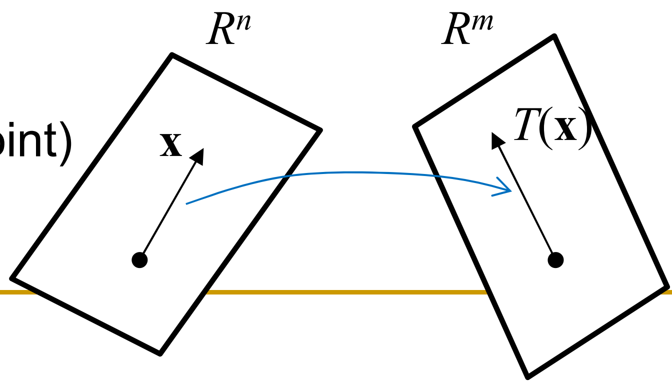
Remark

- A matrix transformation maps linear combinations of vectors in R^n into the corresponding linear combinations in R^m in the sense that

$$T_A(k_1\mathbf{u}_1+k_2\mathbf{u}_2+\ldots+k_r\mathbf{u}_r) = k_1T_A(\mathbf{u}_1)+k_2T_A(\mathbf{u}_2)+\ldots+k_rT_A(\mathbf{u}_r)$$

- Depending on whether n-tuples and m-tuples are regarded as vectors or points, the geometric effect of a matrix transformation $T_A:R^n \rightarrow R^m$ is to map each vector (point) in R^n into a vector in R^m

x is a vector (or point)



$T(\mathbf{x})$ is a vector (or point)

Theorem 4.9.2

- If $T_A: R^n \rightarrow R^m$ and $T_B: R^n \rightarrow R^m$ are matrix transformations, and if $T_A(\mathbf{x}) = T_B(\mathbf{x})$ for every vector $\mathbf{x}$ in R^n, then $A=B$.
- Proof:
  - To say that $T_A(\mathbf{x}) = T_B(\mathbf{x})$ for every vector $\mathbf{x}$ in R^n is the same as saying that $A\mathbf{x} = B\mathbf{x}$ for every vector $\mathbf{x}$ in R^n.
  - This is true, in particular, if $\mathbf{x}$ is any of **the standard basis vectors** $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ for R^n; that is $A\mathbf{e}_j = B\mathbf{e}_j$ ($j=1,2,\ldots,n$)
  - Since every entry of $\mathbf{e}_j$ is 0 except for the jth, which is 1, it follows from Theorem 1.3.1 that $A\mathbf{e}_j$ is the jth column of A, and $B\mathbf{e}_j$ is the jth column of B. Therefore, $A = B$.

Zero Transformation

- Zero Transformation from R^n to R^m
  - If 0 is the $m \times n$ zero matrix and $\mathbf{0}$ is the zero vector in R^n, then for every vector $\mathbf{x}$ in R^n

$$T_0(\mathbf{x}) = 0\mathbf{x} = \mathbf{0}$$

  - So multiplication by zero maps every vector in R^n into the **zero vector** in R^m. We call T_0 **the zero transformation** from R^n to R^m.

Identity Operator

- Identity Operator on R^n
  - If I is the $n \times n$ identity, then for every vector $\mathbf{x}$ in R^n

$$T_I(\mathbf{x}) = I\mathbf{x} = \mathbf{x}$$

  - So multiplication by I maps every vector in R^n into itself.
  - We call T_I the identity operator on R^n.

A Procedure for Finding Standard Matrices

- To find the standard matrix A for a matrix transformations from R^n to R^m:
- $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard basis vectors for R^n.
- Suppose that the images of these vectors under the transformation T_A are

$$T_A(\mathbf{e}_1) = A\mathbf{e}_1,\ T_A(\mathbf{e}_2) = A\mathbf{e}_2,\ \ldots,\ T_A(\mathbf{e}_n) = A\mathbf{e}_n$$

- $A\mathbf{e}_j$ is just the jth column of the matrix A, Thus,

$$A = [T] = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2) \mid \ldots \mid T(\mathbf{e}_n)]$$

Reflection Operators

- In general, operators on R^2 and R^3 that map each vector into its symmetric image about some line or plane are called reflection (倒影) operators.
- Such operators are linear.

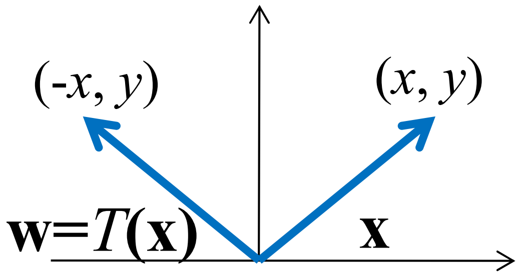
Example

- If we let $\mathbf{w}$=$T(\mathbf{x})$, then the equations relating the components of $\mathbf{x}$ and $\mathbf{w}$ are

$$w_1 = -x = -x + 0y$$
$$w_2 = y = 0x + y$$

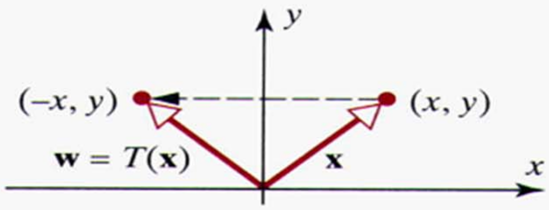
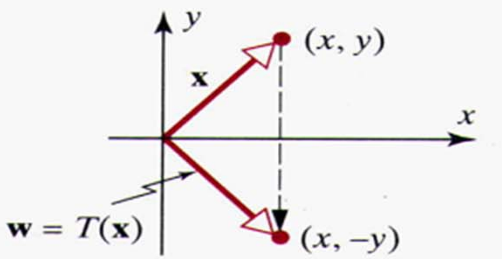
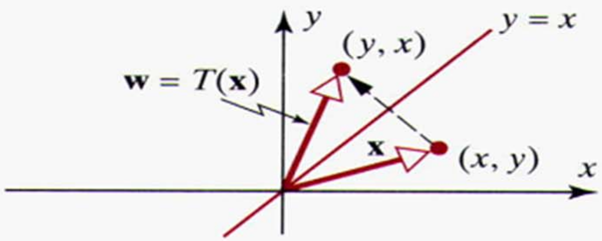
or, in matrix form

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

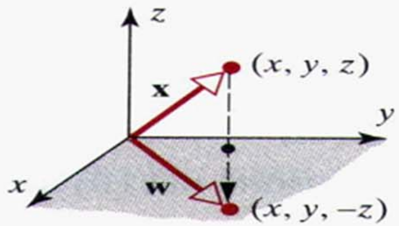
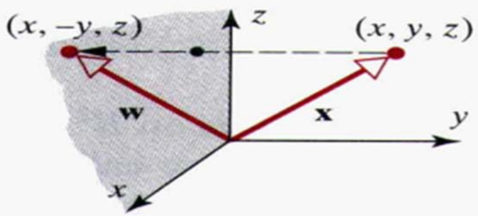
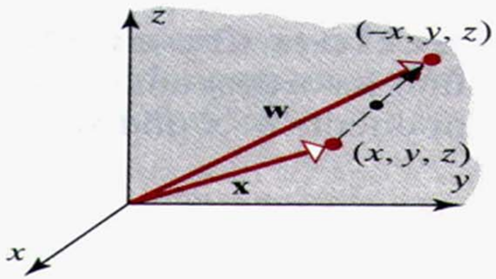


- The standard matrix for T is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Reflection Operators (2-Space)

Operator	Illustration	Equations	Standard Matrix
Reflection about the y-axis		$w_1 = -x$ $w_2 = y$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection about the x-axis		$w_1 = x$ $w_2 = -y$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about the line $y = x$		$w_1 = y$ $w_2 = x$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Reflection Operators (3-Space)

Operator	**Illustration**	**Equations**	**Standard Matrix**
Reflection about the xy-plane		$w_1 = x$ $w_2 = y$ $w_3 = -z$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Reflection about the xz-plane		$w_1 = x$ $w_2 = -y$ $w_3 = z$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Reflection about the yz-plane		$w_1 = -x$ $w_2 = y$ $w_3 = z$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Projection Operators

- In general, a projection operator (or more precisely an orthogonal projection operator) on R^2 or R^3 is any operator that maps each vector into its orthogonal projection on a line or plane through the origin.
- The projection operators are linear.

Example

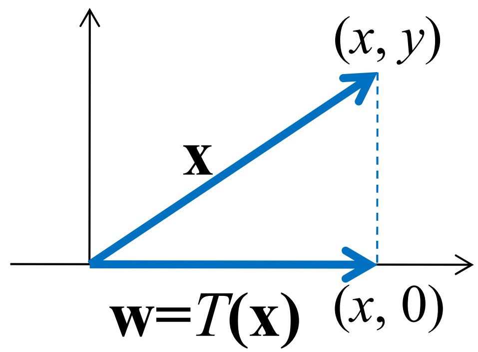
- Consider the operator T: $R^2 \rightarrow R^2$ that maps each vector into its orthogonal projection on the x-axis. The equations relating the components of $\mathbf{x}$ and $\mathbf{w}=T(\mathbf{x})$ are

$$w_1 = x = x + 0y$$
$$w_2 = 0 = 0x + 0y$$

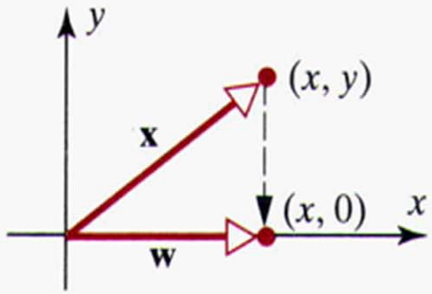
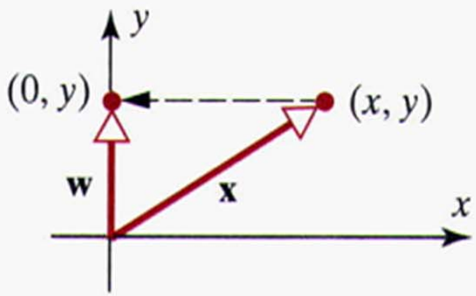
or, in matrix form

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

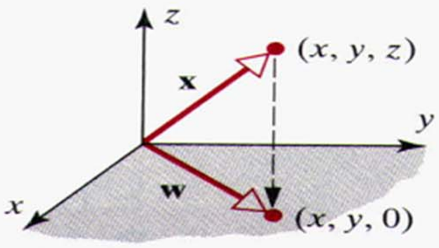
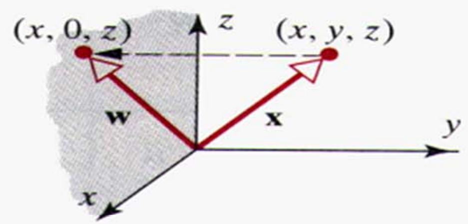
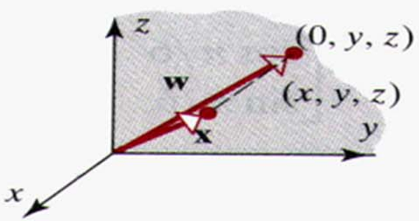
- The standard matrix for T is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$



Projection Operators

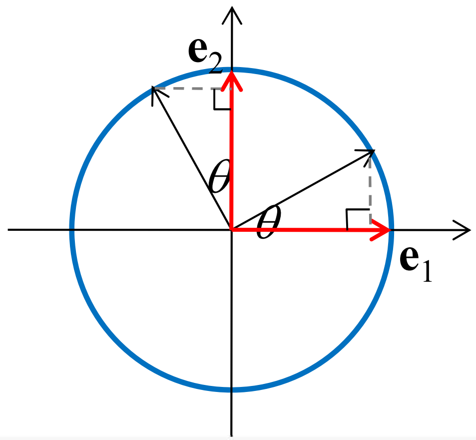
Operator	Illustration	Equations	Standard Matrix
Orthogonal projection on the x-axis		$w_1 = x$ $w_2 = 0$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Orthogonal projection on the y-axis		$w_1 = 0$ $w_2 = y$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

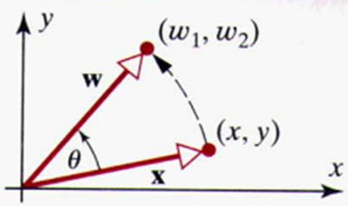
Projection Operators

Operator	Illustration	Equations	Standard Matrix
Orthogonal projection on the xy-plane		$w_1 = x$ $w_2 = y$ $w_3 = 0$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Orthogonal projection on the xz-plane		$w_1 = x$ $w_2 = 0$ $w_3 = z$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Orthogonal projection on the yz-plane		$w_1 = 0$ $w_2 = y$ $w_3 = z$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rotation Operators

- **The rotation operator $T{:}R^2 \rightarrow R^2$ moves points counterclockwise about the origin through an angle θ**
- Find the standard matrix
- $T(\mathbf{e}_1) = T(1,0) = (\cos\theta, \sin\theta)$
- $T(\mathbf{e}_2) = T(0,1) = (-\sin\theta, \cos\theta)$



Operator	Illustration	Equations	Standard Matrix
Rotation through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Example

- If each vector in R^2 is rotated through an angle of $\pi/6$ (30°) ,then the image **w** of a vector

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

is

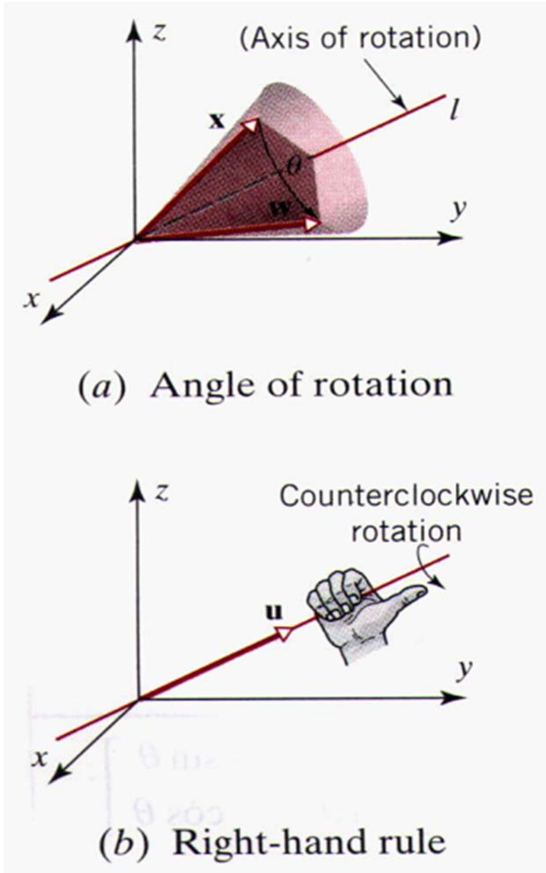
$$\mathbf{w} = \begin{bmatrix} \cos \pi/6 & -\sin \pi/6 \\ \sin \pi/6 & \cos \pi/6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2\, x - 1/2\, y \\ 1/2\, x + \sqrt{3}/2\, y \end{bmatrix}$$

- For example, the image of the vector

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{is} \quad \mathbf{w} = \begin{bmatrix} \dfrac{\sqrt{3} - 1}{2} \\ \dfrac{1 + \sqrt{3}}{2} \end{bmatrix}$$

A Rotation of Vectors in R^3

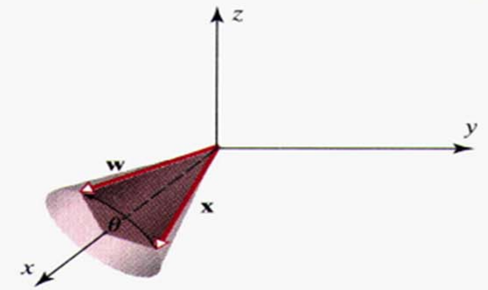
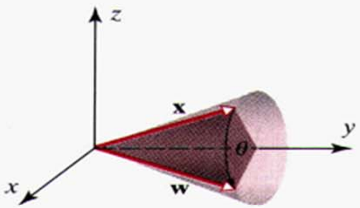
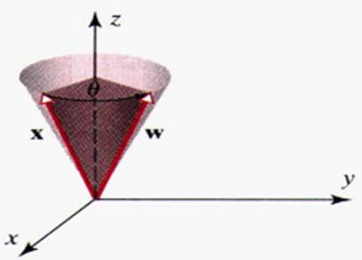
- A rotation of vectors in R^3 is usually described in relation to a ray emanating from (發源自) the origin, called the axis of rotation.
- As a vector revolves around the axis of rotation it sweeps out some portion of a cone (圓錐體).
- The angle of rotation is described as "clockwise" or "counterclockwise" in relation to a viewpoint that is along the axis of rotation *looking toward the origin.*
- The axis of rotation can be specified by a nonzero vector **u** that runs along the axis of rotation and has its initial point at the origin.
- The counterclockwise direction for a rotation about its axis can be determined by a "right-hand rule".



(a) Angle of rotation

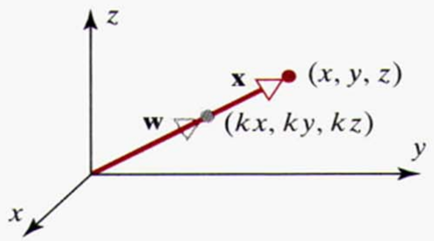
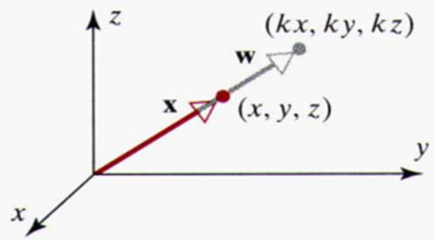
(b) Right-hand rule

A Rotation of Vectors in R^3

Operator	Illustration	Equations	Standard Matrix
Counterclockwise rotation about the positive x-axis through an angle θ		$w_1 = x$ $w_2 = y\cos\theta - z\sin\theta$ $w_3 = y\sin\theta + z\cos\theta$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive y-axis through an angle θ		$w_1 = x\cos\theta + z\sin\theta$ $w_2 = y$ $w_3 = -x\sin\theta + z\cos\theta$	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive z-axis through an angle θ		$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$ $w_3 = z$	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Dilation and Contraction Operators

- If k is a nonnegative scalar, the operator on R^2 or R^3 is called a contraction with factor k if $0 \leq k \leq 1$ (以因素k收縮) and a dilation with factor k if $k \geq 1$ (以因素k膨脹).

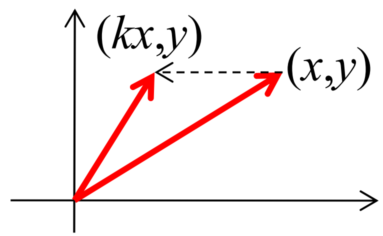
Operator	Illustration	Equations	Standard Matrix
Contraction with factor k on R^3 $(0 \leq k \leq 1)$		$w_1 = kx$ $w_2 = ky$ $w_3 = kz$	$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$
Dilation with factor k on R^3 $(k \geq 1)$		$w_1 = kx$ $w_2 = ky$ $w_3 = kz$	

Compression or Expansion

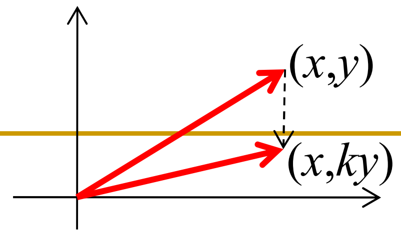
- If T: $R^2 \rightarrow R^2$ is a compression ($0<k<1$) or expansion ($k>1$) in the x-direction with factor k, then

$$T(\boldsymbol{e}_1) = T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}k\\0\end{bmatrix} \qquad T(\boldsymbol{e}_2) = T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$$

so the standard matrix for T is $\begin{bmatrix}k & 0\\0 & 1\end{bmatrix}$.

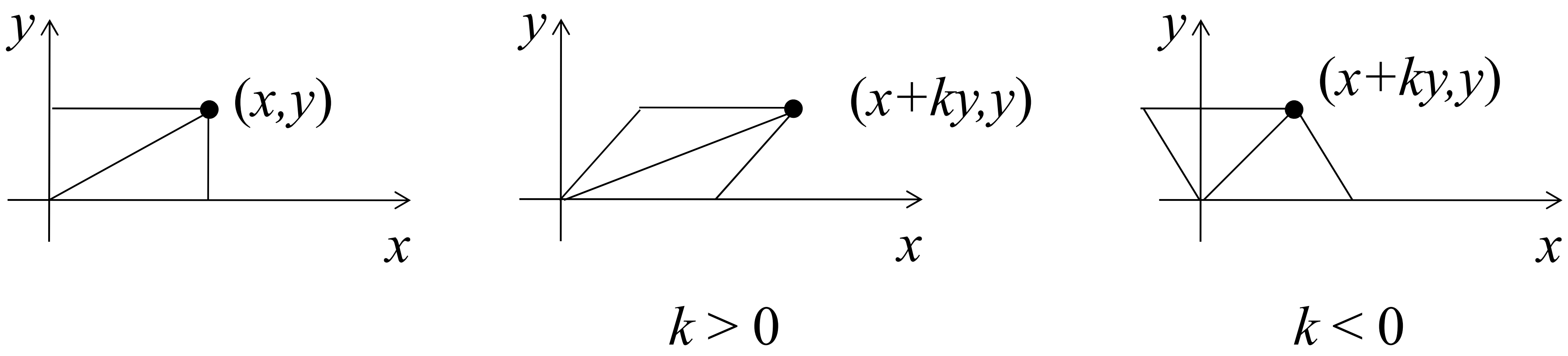


- Similarly, the standard matrix for a compression or expansion in the y-direction is $\begin{bmatrix}1 & 0\\0 & k\end{bmatrix}$



Shears

- **A shear (剪) in the *x*-direction with factor *k*** is a transformation that moves each point (x,y) parallel to the x-axis by an amount ky to the new position $(x+ky,y)$.
- Points farther from the x-axis move a greater distance than those closer.



Shears

- If $T: R^2 \rightarrow R^2$ is a shear with factor k in the x-direction, then

$$T(\boldsymbol{e}_1) = T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}x+ky\\y\end{bmatrix} = \begin{bmatrix}1+k0\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

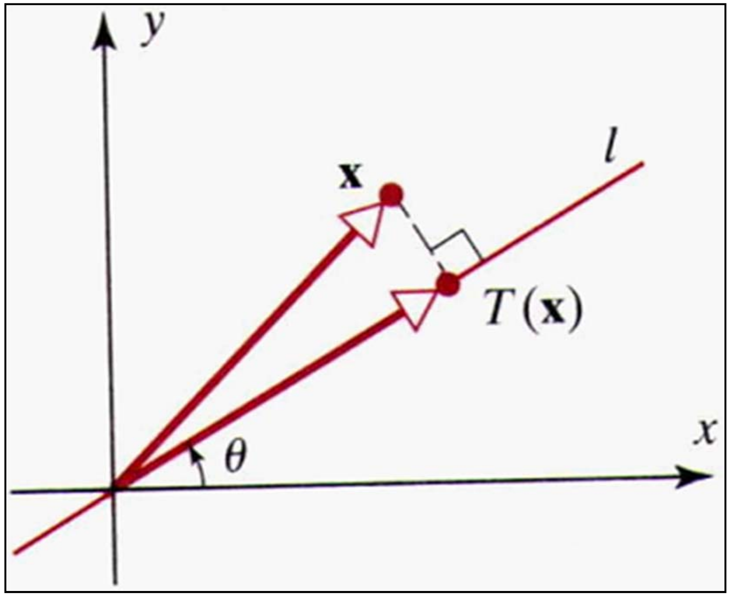
$$T(\boldsymbol{e}_2) = T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}x+ky\\y\end{bmatrix} = \begin{bmatrix}0+k1\\1\end{bmatrix} = \begin{bmatrix}k\\1\end{bmatrix}$$

- The standard matrix for T is $\begin{bmatrix}1 & k\\0 & 1\end{bmatrix}$

- Similarly, the standard matrix for a shear in the y-direction with factor k is $\begin{bmatrix}1 & 0\\k & 1\end{bmatrix}$

Example (Standard Matrix for a Projection Operator)

- Let l be the line in the xy-plane that passes through the origin and makes an angle θ with the positive x-axis, where $0 \leq \theta \leq \pi$. Let T: $R^2 \rightarrow R^2$ be a linear operator that maps each vector into orthogonal projection on l.
  - Find the standard matrix for T.
  - Find the orthogonal projection of the vector $\mathbf{x} = (1,5)$ onto the line through the origin that makes an angle of $\theta = \pi/6$ with the positive x-axis.



Example

- The standard matrix for T can be written as

$$[T] = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2)]$$

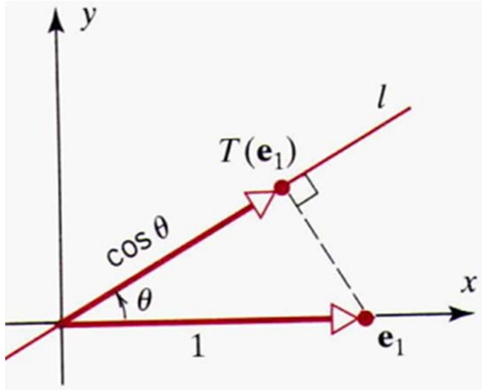
- Consider the case $0 \le \theta \le \pi/2$.
  - $\|T(\mathbf{e}_1)\| = \cos\theta$

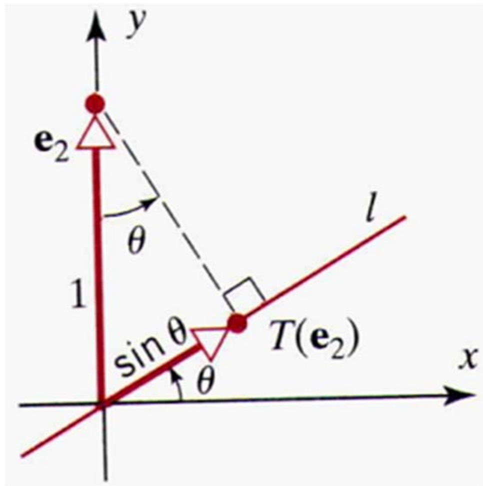
$$\Rightarrow \quad T(\mathbf{e}_1) = \begin{bmatrix} \|T(\mathbf{e}_1)\|\cos\theta \\ \|T(\mathbf{e}_1)\|\sin\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta \\ \sin\theta\cos\theta \end{bmatrix}$$

  - $\|T(\mathbf{e}_2)\| = \sin\theta$

$$\Rightarrow \quad T(\mathbf{e}_2) = \begin{bmatrix} \|T(\mathbf{e}_2)\|\cos\theta \\ \|T(\mathbf{e}_2)\|\sin\theta \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\theta \\ \sin^2\theta \end{bmatrix}$$

$$\Rightarrow \quad [T] = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$





Example

$$[T]=\begin{bmatrix}\cos^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{bmatrix}$$

- Since sin (π/6) = 1/2 and cos (π/6) = $\sqrt{3}$/2, it follows from part (a) that the standard matrix for this projection operator is

$$[T]=\begin{bmatrix}3/4 & \sqrt{3}/4\\ \sqrt{3}/4 & 1/4\end{bmatrix}$$

Thus,

$$T\left(\begin{bmatrix}1\\5\end{bmatrix}\right)=\begin{bmatrix}3/4 & \sqrt{3}/4\\ \sqrt{3}/4 & 1/4\end{bmatrix}\begin{bmatrix}1\\5\end{bmatrix}=\begin{bmatrix}\dfrac{3+5\sqrt{3}}{4}\\ \dfrac{\sqrt{3}+5}{4}\end{bmatrix}$$

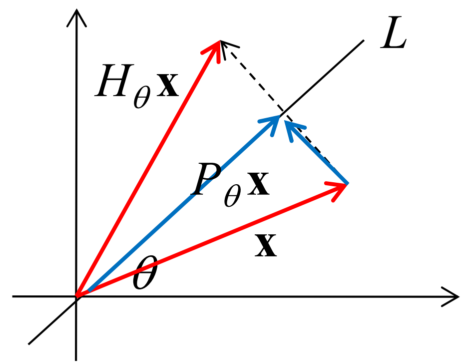
Reflections About Lines Through the Origin

- Let P_θ denote the standard matrix of orthogonal projections on lines through the origin

$P_\theta\mathbf{x} - \mathbf{x} = (1/2)(H_\theta\,\mathbf{x} - \mathbf{x})$, or equivalently $H_\theta\,\mathbf{x} = (2\,P_\theta - I)\mathbf{x}$

- $H_\theta = (2\,P_\theta - I)$

$$H_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$



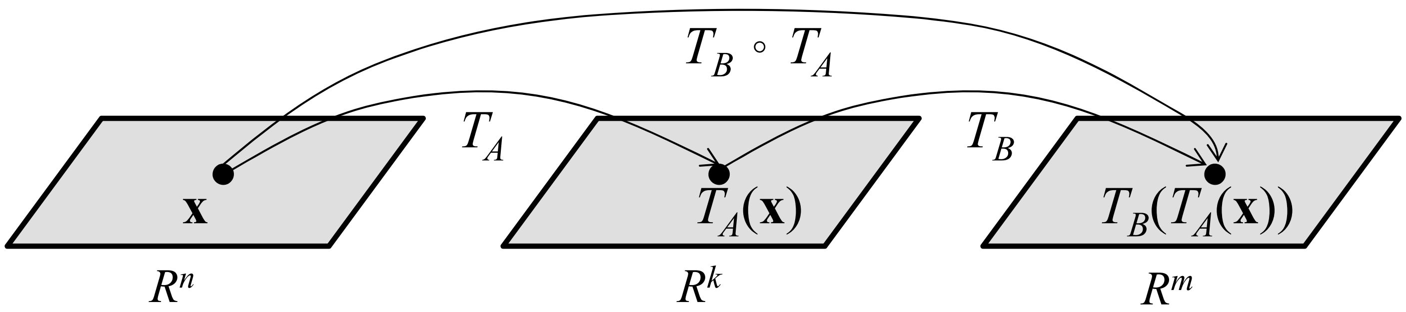
4.10
Properties of Matrix Transformations

Composition of T_B with T_A

- Definition
  - If $T_A : R^n \to R^k$ and $T_B : R^k \to R^m$ are linear transformations, *the composition of T_B with T_A*, denoted by $T_B \circ T_A$ (read "T_B circle T_A"), is the function defined by the formula

$$(T_B \circ T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x}))$$

where $\mathbf{x}$ is a vector in R^n.



Composition of T_B with T_A

- This composition is itself a matrix transformation since

$$(T_B \circ T_A)(\mathbf{x}) = (T_B(T_A(\mathbf{x})) = B(T_A(\mathbf{x})) = B(A\mathbf{x}) = (BA)\mathbf{x}$$

- It is multiplication by BA, i.e. $T_B \circ T_A = T_{BA}$
- The compositions can be defined for more than two linear transformations.
- For example, if $T_1 : U \to V$ and $T_2 : V \to W$,and $T_3 : W \to Y$ are linear transformations, then the composition $T_3 \circ T_2 \circ T_1$ is defined by $(T_3 \circ T_2 \circ T_1)(\mathbf{u}) = T_3(T_2(T_1(\mathbf{u})))$

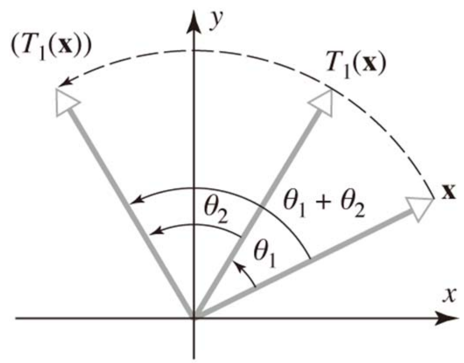
Remark

- It is not true, in general, that $AB = BA$
- So it is not true, in general, that $T_B \circ T_A = T_A \circ T_B$

Example

- Let $T_1:R^2 \to R^2$ and $T_2:R^2 \to R^2$ be the matrix operators that rotate vectors through the angles θ_1 and θ_2, respectively.
- The operation $(T_2 \circ T_1)(\mathbf{x})=T_2(T_1(\mathbf{x}))$ first rotates $\mathbf{x}$ through the angle θ_1, then rotates $T_1(\mathbf{x})$ through the angle θ_2.

$$[T_1] = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \qquad [T_2] = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$



$$[T_2 \circ T_1] = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$$

$$[T_2][T_1] = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_2\cos\theta_1 - \sin\theta_2\sin\theta_1 & -(\cos\theta_2\sin\theta_1 + \sin\theta_2\cos\theta_1) \\ \sin\theta_2\cos\theta_1 + \cos\theta_2\sin\theta_1 & -\sin\theta_2\sin\theta_1 + \cos\theta_2\cos\theta_1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix} = [T_2 \circ T_1]$$

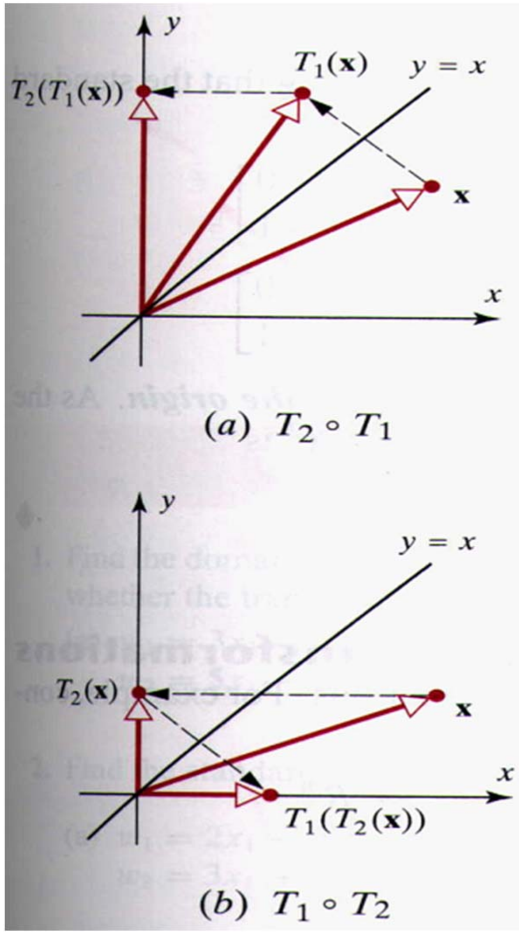
Composition is Not Commutative

- Let T_1 be the reflection operator
- Let T_2 be the orthogonal projection on the y-axis

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

so $[T_1 \circ T_2] \neq [T_2 \circ T_1]$



Composition of Two Reflections

- Let T_1 be the reflection about the y-axis, and let T_2 be the reflection about the x-axis. In this case, $T_1 \circ T_2$ and $T_2 \circ T_1$ are the same.

$$(T_1 \circ T_2)(x, y) = T_1(x, -y) = (-x, -y)$$

$$(T_2 \circ T_1)(x, y) = T_2(-x, y) = (-x, -y)$$

$$[T_1 \circ T_2] = [T_1]\,[T_2] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$[T_2 \circ T_1] = [T_2]\,[T_1] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

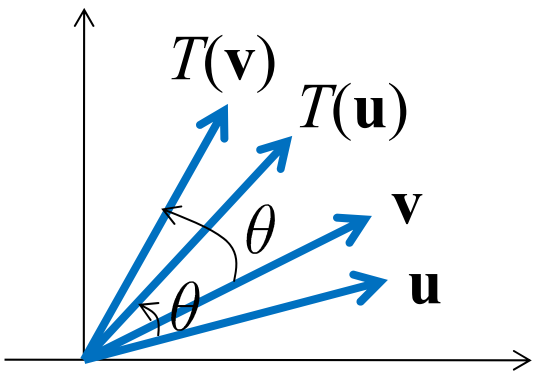
One-to-One Linear transformations

- Definition
  - A linear transformation $T : R^n \to R^m$ is said to be one-to-one if T maps distinct vectors (points) in R^n into distinct vectors (points) in R^m

- Remark:
  - That is, for each vector $\mathbf{w}$ in the range of a one-to-one linear transformation T, there is exactly one vector $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{w}$.

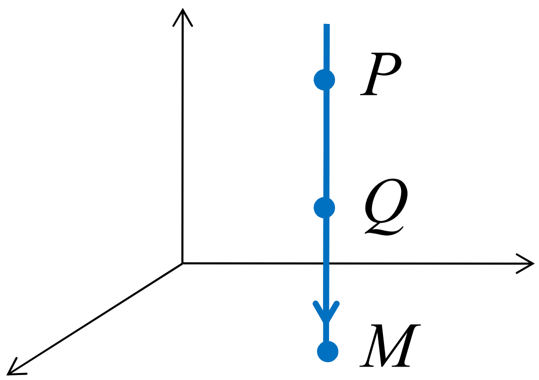
Example

One-to-one linear transformation



Distinct vectors **u** and **v** are rotated into distinct vectors $T(\mathbf{u})$ and $T(\mathbf{v})$.

Not one-to-one linear transformation



The distinct points P and Q are mapped into the same point M.

Theorem 4.10.1 (Equivalent Statements)

- If A is an $n \times n$ matrix and $T_A : R^n \rightarrow R^n$ *is* multiplication by A, then the following statements are equivalent.
  - A is invertible
  - The range of T_A is R^n
  - T_A is one-to-one

$$A\mathbf{x}=\mathbf{b}$$

Proof of Theorem 4.10.1

- (a)→(b): Assume A is invertible. $A\mathbf{x}=\mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$ in R^n. This implies that T_A maps $\mathbf{x}$ into the arbitrary vector $\mathbf{b}$ in R^n, which implies the range of T_A is R^n.
- (b)→(c): Assume the range of T_A is R^n. For every vector $\mathbf{b}$ in R^n there is some vector $\mathbf{x}$ in R^n for which $T_A(\mathbf{x})=\mathbf{b}$ and hence the linear system $A\mathbf{x}=\mathbf{b}$ is consistent for every vector $\mathbf{b}$ in R^n. But we know $A\mathbf{x}=\mathbf{b}$ has a unique solution, and hence for every vector $\mathbf{b}$ in the range of T_A there is exactly one vector $\mathbf{x}$ in R^n such that $T_A(\mathbf{x})=\mathbf{b}$.

Example

- The rotation operator $T : R^2 \to R^2$ is one-to-one
  - The standard matrix for T is

$$[T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

  - $[T]$ is invertible since

$$\det\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1 \neq 0$$

Example

- The projection operator $T : R^3 \rightarrow R^3$ is **not** one-to-one
  - The standard matrix for T is
  $$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
  - $[T]$ is not invertible since $\det[T] = 0$

Inverse of a One-to-One Linear Operator

- Suppose $T_A : R^n \to R^n$ is a one-to-one linear operator

  $\Rightarrow$ The matrix A is invertible.

  $\Rightarrow$ $T_{A^{-1}} : R^n \to R^n$ is itself a linear operator; it is called the inverse of T_A.

  $$\Rightarrow \quad T_A(T_{A^{-1}}(\mathbf{x})) = AA^{-1}\mathbf{x} = I\mathbf{x} = \mathbf{x} \text{ and}$$
  $$T_{A^{-1}}(T_A(\mathbf{x})) = A^{-1}A\mathbf{x} = I\mathbf{x} = \mathbf{x}$$

  $$\Rightarrow \quad T_A \circ T_{A^{-1}} = T_{AA^{-1}} = T_I \quad \text{and}$$
  $$T_{A^{-1}} \circ T_A = T_{A^{-1}A} = T_I$$

Inverse of a One-to-One Linear Operator

- If **w** is the image of **x** under T_A, then T_A^{-1} maps **w** back into **x**, since

$$T_A^{-1}(\mathbf{w}) = T_A^{-1}(T_A(\mathbf{x})) = \mathbf{x}$$

- When a one-to-one linear operator on R^n is written as $T : R^n \to R^n$, then the inverse of the operator T is denoted by T^{-1}.
- Thus, by the standard matrix, we have $[T^{-1}]=[T]^{-1}$

Example

- Let $T : R^2 \to R^2$ be the operator that rotates each vector in R^2 through the angle θ:

$$[T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- Undo the effect of T means rotate each vector in R^2 through the angle -θ.

- This is exactly what the operator T^{-1} does: the standard matrix T^{-1} is

$$[T^{-1}] = [T]^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

- The only difference is that the angle θ is replaced by -θ

Example

- Show that the linear operator $T : R^2 \rightarrow R^2$ defined by the equations

$$w_1 = 2x_1 + x_2$$

$$w_2 = 3x_1 + 4x_2$$

is one-to-one, and find $T^{-1}(w_1, w_2)$.

- Solution:

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow [T] = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \Rightarrow [T^{-1}] = [T]^{-1} = \begin{bmatrix} \frac{4}{5} & -\frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

$$\Rightarrow [T^{-1}] \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} w_1 - \frac{1}{5} w_2 \\ -\frac{3}{5} w_1 + \frac{2}{5} w_2 \end{bmatrix}$$

$$\Rightarrow T^{-1}(w_1, w_2) = \left(\frac{4}{5} w_1 - \frac{1}{5} w_2, -\frac{3}{5} w_1 + \frac{2}{5} w_2 \right)$$

Linearity Properties

- Theorem 4.10.2 (Properties of Linear Transformations)
  - A transformation $T : R^n \rightarrow R^m$ is linear if and only if the following relationships hold for all vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n and every scalar c.
    - $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
    - $T(c\mathbf{u}) = cT(\mathbf{u})$

Proof of Theorem 4.10.2

- (=>) Follow from Theorem 4.9.1
- (<=) Conversely, assume that properties (a) and (b) hold for the transformation T. We can prove that T is linear by finding a matrix A with the property that $T(\mathbf{x}) = A\mathbf{x}$ for all vectors $\mathbf{x}$ in R^n.
- The property (a) can be extended to three or more terms. $T(\mathbf{u}+\mathbf{v}+\mathbf{w}) = T(\mathbf{u}+(\mathbf{v}+\mathbf{w})) = T(\mathbf{u})+T(\mathbf{v}+\mathbf{w}) = T(\mathbf{u})+T(\mathbf{v})+T(\mathbf{w})$
- More generally, for any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in R^n, we have

$$T(\mathbf{v}_1+\mathbf{v}_2+\ldots+\mathbf{v}_k) = T(\mathbf{v}_1) + T(\mathbf{v}_2) + \ldots + T(\mathbf{v}_k)$$

Proof of Theorem 4.10.2

- Now, to find the matrix A, let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be the vectors

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots\cdots \quad \boldsymbol{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

- Let A be the matrix whose successive column vectors are $T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)$; that is
$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2) \mid \ldots \mid T(\mathbf{e}_n)]$

Proof of Theorem 4.10.2

- If $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is any vector in R^n, then as discussed in Section 1.3 (Theorem 1.3.1), the product $A\mathbf{x}$ is a linear combination of the column vectors of A with coefficients $\mathbf{x}$, so

$$
\begin{aligned}
A\mathbf{x} &= x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) + \ldots + x_nT(\mathbf{e}_n) \\
&= T(x_1\mathbf{e}_1) + T(x_2\mathbf{e}_2) + \ldots + T(x_n\mathbf{e}_n) \\
&= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + .. + x_n\mathbf{e}_n) \\
&= T(\mathbf{x})
\end{aligned}
$$

Theorem 4.10.3

- **Every linear transformation from R^n to R^m is a matrix transformation, and conversely, every matrix transformation from R^n to R^m is a linear transformation.**

Theorem 4.10.4 (Equivalent Statements)

- If A is an $m \times n$ matrix, and if $T_A : R^n \rightarrow R^n$ is multiplication by A, then the following are equivalent:
  - A is invertible.
  - $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
  - The reduced row-echelon form of A is I_n.
  - A is expressible as a product of elementary matrices.
  - $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
  - $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
  - $\det(A) \neq 0$.
  - The column vectors of A are linearly independent.
  - The row vectors of A are linearly independent.
  - The column vectors of A span R^n.
  - The row vectors of A span R^n.
  - The column vectors of A form a basis for R^n.
  - The row vectors of A form a basis for R^n.
  - A has rank n.
  - A has nullity 0.
  - The orthogonal complement of the nullspace of A is R^n.
  - The orthogonal complement of the row space of A is $\{\mathbf{0}\}$.
  - The range of T_A is R^n.
  - T_A is one-to-one.

4.11
Geometry of Matrix Operations

Example: Transforming with Diagonal Matrices

- Suppose that the xy-plane first is compressed or expanded by a factor of k_1 in the x-direction and then is compressed or expanded by a factor of k_2 in the y-direction. Find a single matrix operator that performs both operations.

$$\begin{bmatrix} k_1 & 0 \\ 0 & 1 \end{bmatrix}$$

x-compression (expansion)

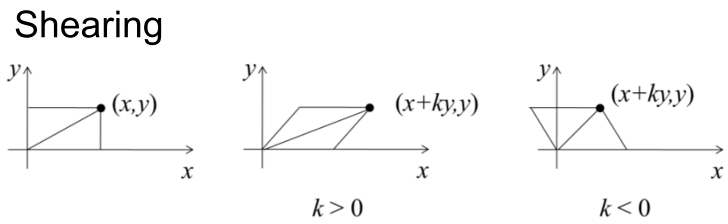
$$\begin{bmatrix} 1 & 0 \\ 0 & k_2 \end{bmatrix}$$

y-compression (expansion)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} k_1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$$

- If $k_1=k_2=k$, this is a contraction (收縮) or dilation (擴張). $A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

Example



- Find a matrix transformation from R^2 to R^2 that first shears by a factor of 2 in the x-direction and then reflects about $y = x$.
- The standard matrix for the shear is $A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

  and for the reflection is $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

- Thus the standard matrix for the **shear** followed by the **reflection** is

$$A_2 A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

Example

- Find a matrix transformation from R^2 to R^2 that first reflects about $y = x$ and then shears by a factor of 2 in the x-direction.

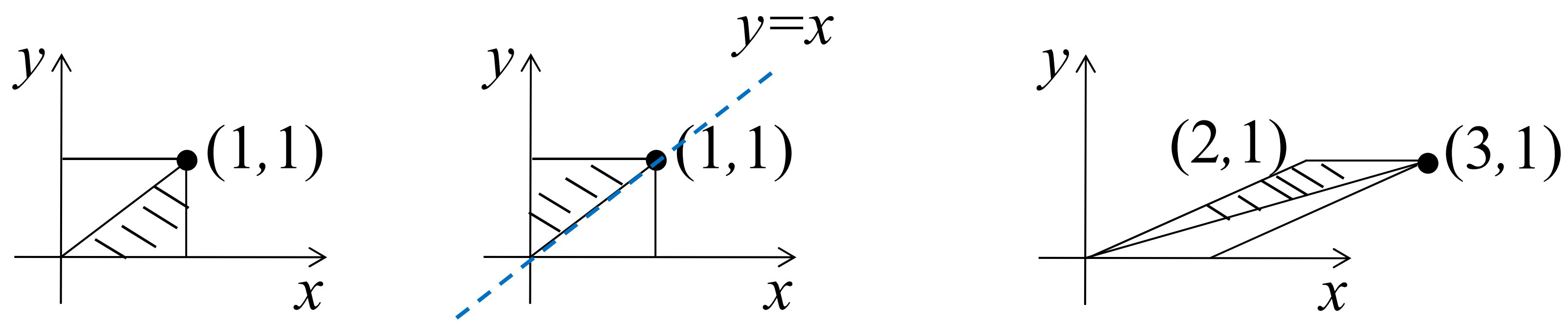
$$A_1 A_2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

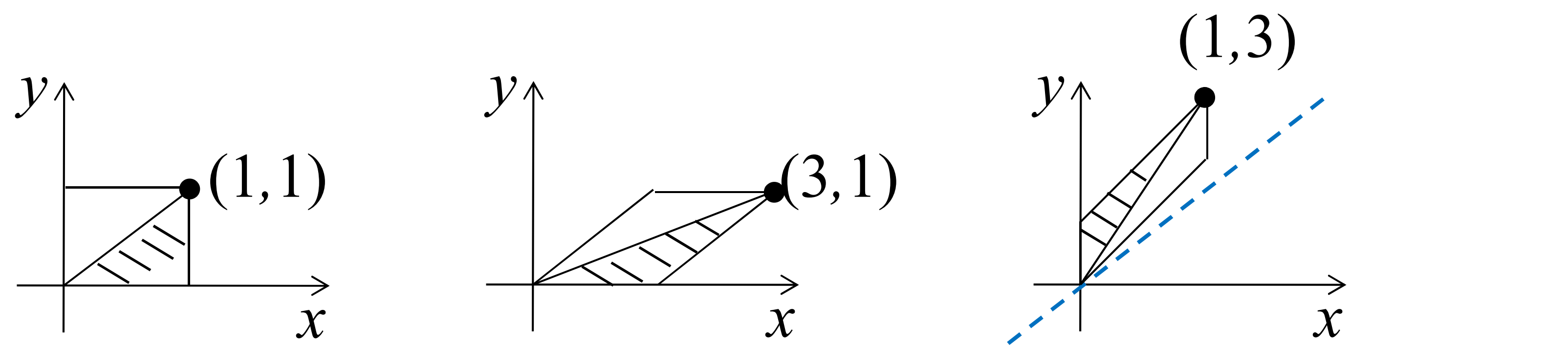
- Note that $A_1 A_2 \neq A_2 A_1$

Geometry

$$A_1A_2 = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}2 & 1\\1 & 0\end{bmatrix}$$

a matrix transformation from R^2 to R^2 that first reflects about $y = x$ and then shears by a factor of 2 in the x-direction





$$A_2A_1 = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 2\end{bmatrix}$$

a matrix transformation from R^2 to R^2 that first shears by a factor of 2 in the x-direction and then reflects about $y = x$

Geometry of One-to-One Matrix Operators

- A matrix transformation T_A is **one-to-one** if and only if A is **invertible** and **can be expressed as a product of elementary matrices**.

$$A = E_1 E_2 \cdots E_r$$
$$T_A = T_{E_1 E_2 \cdots E_r} = T_{E_1} \circ T_{E_2} \circ \cdots \circ T_{E_r}$$

- **Theorem 4.11.1:** If E is an elementary matrix, then T_E: $R^2 \rightarrow R^2$ is one of the following:
  - A shear along a coordinate axis
  - A reflection about y=x
  - A compression along a coordinate axis
  - An expansion along a coordinate axis
  - A reflection about a coordinate axis
  - A compression or expansion along a coordinante axis followed by a reflection about a coordinate axis

Proof of Theorem 4.11.1

- Because a 2×2 elementary matrix results from performing **a single elementary row operation** on the 2×2 identity matrix, it must have one of the following forms:

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

- $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ represent **shears** along coordinates axes.
- $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ represents a **reflection** about $y = x$.

Proof of Theorem 4.11.1

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

- If $k > 0$, $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ represent compressions or expansion along coordinate axes, depending on whether $0 \le k \le 1$ (compression) or $k \ge 1$ (expansion).
- If ***k* < 0**, and if we express k in the form k=-k_1, where $\mathbf{k_1>0}$, then

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -k_1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -k_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & k_1 \end{bmatrix}$$

Proof of Theorem 4.11.1

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -k_1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 & 0 \\ 0 & 1 \end{bmatrix}$$

- It represents a **compression or expansion** along the x-axis followed by a **reflection** (倒影) about the y-axis.

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -k_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & k_1 \end{bmatrix}$$

- It represents a **compression or expansion** along the y-axis followed by a **reflection** about the x-axis.

Theorem 4.11.2

- **If T_A:$R^2 \rightarrow R^2$ is multiplication by an invertible matrix A, then the geometric effect of T_A is the same as an appropriate succession of shears, compressions, expansions, and reflections.**

Example: Geometric Effect of Multiplication by a Matrix

- Assuming that k_1 and k_2 are positive, express the diagonal matrix $A = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$ as a product of elementary matrices, and describe the geometric effect of multiplication by A in terms of compressions and expansions.
- We know

interchangeable!

$$A = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} k_1 & 0 \\ 0 & 1 \end{bmatrix}$$

which shows the geometric effect of compressing or expanding by a factor of k_1 in the x-direction and then compressing or expanding by a factor of k_2 in the y-direction.

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

- Express A as a product of elementary matrices, and then describe the geometric effect of multiplication by A in terms of shears, compressions, expansion, and reflections.
- A can be reduced to I as follows:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Add -3 times the first row to the second

Multiply the second row by -1/2

Add -2 times the second row to the first

- The three successive row operations can be performed by multiplying on the left successively by

$$E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$E_3E_2E_1A=I \quad \Rightarrow A=E_1^{-1}E_2^{-1}E_3^{-1}$$

Example

interchangeable!

- Inverting these matrices

4   3, 2   1

$$A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

- Reading from right to left and noting that

$$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

it follows that the effect of multiplying by A is equivalent to

1. shearing by a factor of 2 in the x-direction, $(x+2y, y)$

interchangeable!

2. then expanding by a factor of 2 in the y-direction, $(x, 2y)$
3. then reflecting about the x-axis, $(x, -y)$
4. then shearing by a factor of 3 in the y-direction. $(x, y+3x)$

Theorem 4.11.3

- If $T{:}R^2 \rightarrow R^2$ is multiplication by an invertible matrix, then
  - (a) the image of a straight line is a straight line.
  - (b) the image of a straight line through the origin is a straight line through the origin.
  - (c) the images of parallel straight lines are parallel straight lines.
  - (d) the images of the line segment joining points P and Q is the line segment joining the images of P and Q.
  - (e) the images of three points lie on a line if and only if the points themselves line on some line.

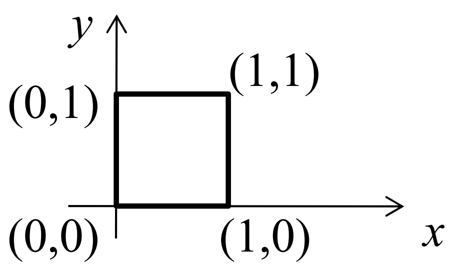
Example: Image of a Square

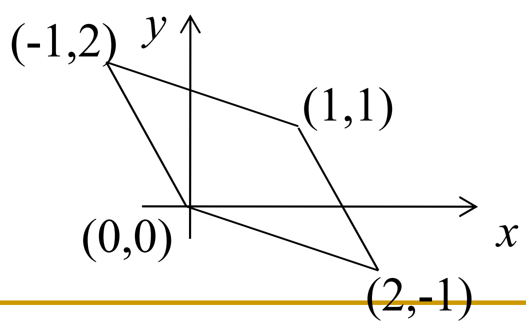
- Sketch **the images of the unit square** under multiplication by

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

- Since $\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$





What is the area of the parallelogram? => [det(*A*)| * area of the original square ?

Example: Image of a Line

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

- The invertible matrix maps the **line *y*=2*x*+1** into another line. Find its equation.
- Let (x,y) be a point on the line y=2x+1, and let (x',y') be its image under multiplication by A. Then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

- So $\begin{aligned} x &= x' - y' \\ y &= -2x' + 3y' \end{aligned} \Rightarrow \begin{aligned} &y = 2x+1 \\ &-2x' + 3y' = 2(x' - y') + 1 \end{aligned} \Rightarrow y' = \frac{4}{5}x' + \frac{1}{5}$
- Thus (x', y') satisfies $y = \frac{4}{5}x + \frac{1}{5}$, which is the equation we want.