Further Topics on Random Variables:

1. Transforms (Moment Generating Functions)
2. Sum of a Random Number of Independent Random Variables

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Reference:

- D. P. Bertsekas, J. N. Tsitsiklis, *Introduction to Probability* , Sections 4.4 & 4.5

Transforms

- Also called **moment generating functions** of random variables

- The **transform** of the distribution of a random variable X is a function $M_X(s)$ of a free parameter s, defined by

$$M_X(s) = \mathbf{E}\left[e^{sX}\right]$$

  - If X is discrete

$$M_X(s) = \sum_x e^{sx} p_X(x)$$

  - If X is continuous

$$M_X(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x) dx$$

Illustrative Examples (1/5)

- **Example 4.22.** Let

$$p_X(x) = \begin{cases} 1/2, & \text{if } x = 2, \\ 1/6, & \text{if } x = 3, \\ 1/3, & \text{if } x = 5. \end{cases}$$

$$\therefore M_X(s) = \mathbf{E}\left[e^{sX}\right] = \sum_x e^{sx} p_X(x)$$

$$= \frac{1}{2} e^{2s} + \frac{1}{6} e^{3s} + \frac{1}{3} e^{5s}$$

Notice that :

$$M_X(0) = \mathbf{E}\left[e^{0X}\right] = \sum_x e^{0x} p_X(x)$$

$$= \sum_x p_X(x) = 1$$

Illustrative Examples (2/5)

- **Example 4.23. The Transform of a Poisson Random Variable.** Consider a Poisson random variable X with parameter λ :

$$p_X(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0,1,2,\ldots$$

$$\begin{aligned}
M_X(s) &= \sum_{x=0}^{\infty} e^{sx} \frac{\lambda^x e^{-\lambda}}{x!} \\
&= e^{-\lambda} \sum_{x=0}^{\infty} \frac{a^x}{x!} \qquad \left(\text{Let } a = e^s \lambda\right) \\
&= e^{-\lambda} e^a \qquad \left(\because \text{McLaurin series} \left(1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots \right) = e^a \right) \\
&= e^{a-\lambda} \\
&= e^{\lambda\left(e^s - 1\right)}
\end{aligned}$$

Illustrative Examples (3/5)

- **Example 4.24. The Transform of an Exponential Random Variable.** Let X be an exponential random variable with parameter λ:

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$\begin{aligned} M_X(s) &= \int_0^\infty e^{sx} \lambda e^{-\lambda x} dx \\ &= \lambda \int_0^\infty e^{(s-\lambda)x} dx \\ &= \lambda \frac{e^{(s-\lambda)x}}{(s-\lambda)} \Bigg|_0^\infty \qquad (\text{if } s - \lambda < 0) \\ &= \frac{\lambda}{\lambda - s} \end{aligned}$$

Notice that :
$M_X(s)$ can be calculated only when $s < \lambda$

Illustrative Examples (4/5)

- **Example 4.25. The Transform of a Linear Function of a Random Variable.** Let $M_X(s)$ be the transform associated with a random variable X. Consider a new random variable $Y = aX + b$. We then have

$$M_Y(s) = \mathbf{E}\left[e^{s(aX+b)}\right] = e^{sb}\mathbf{E}\left[e^{saX}\right] = e^{sb}M_X(sa)$$

  - For example, if X is exponential with parameter $\lambda = 1$ and $Y = 2X + 3$, then

$$M_X(s) = \frac{\lambda}{\lambda - s} = \frac{1}{1-s}$$

$$M_Y(s) = e^{3s}M_X(2s) = e^{3s}\frac{1}{1-2s}$$

Illustrative Examples (5/5)

- **Example 4.26. The Transform of a Normal Random Variable.** Let X be normal with mean μ and variance σ^2.

We first calculate the transform of a standard normal random variable Y

$$f_Y(y)=\frac{1}{\sqrt{2\pi}}e^{-y^2/2}$$

$$M_Y(s)=\int_{-\infty}^{\infty}e^{sy}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy$$

$$=e^{s^2/2}\cdot\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\left[\left(y^2/2\right)-sy+\left(s^2/2\right)\right]}dy$$

$$=e^{s^2/2}\cdot\underset{1}{\boxed{\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-(y-s)^2/2}dy}}$$

$$=e^{s^2/2}$$

Since we also know that $Y=\dfrac{X\text{-}\mu}{\sigma}$,

we can have $X=\sigma Y+\mu$

$$\therefore\ M_X(s)=e^{s\mu}M_Y(s\sigma)$$

$$=e^{s\mu}\cdot e^{s^2\sigma^2/2}$$

$$=e^{s\mu+\left(s^2\sigma^2/2\right)}$$

From Transforms to Moments (1/2)

- Given a random variable X , we have

$$M_X(s)=\mathbf{E}\left[e^{sx}\right]=\int_{-\infty}^{\infty} e^{sx} f_X(x)dx$$ (If X is continuous)

Or

$$M_X(s)=\mathbf{E}\left[e^{sx}\right]=\sum_x e^{sx} p_X(x)$$ (If X is discrete)

- When taking the derivative of the above functions with respect to s (for example, the continuous case)

$$\frac{dM_X(s)}{ds}=\frac{d\int_{-\infty}^{\infty} e^{sx} f_X(x)dx}{ds}=\int_{-\infty}^{\infty} x e^{sx} f_X(x)dx$$

  - If we evaluate it at $s=0$, we can further have

the first moment of X

$$\frac{dM_X(s)}{ds}\bigg|_{s=0}=\int_{-\infty}^{\infty} x e^{sx} f_X(x)dx\bigg|_{s=0}=\int_{-\infty}^{\infty} x f_X(x)dx=\mathbf{E}[x]$$

From Transforms to Moments (2/2)

- More generally, taking the differentiation of $M_X(s)$ *n* times with respect to s will yield

$$\frac{d^n M_X(s)}{d^n s}\Big|_{s=0} = \int_{-\infty}^{\infty} x^n e^{sx} f_X(x) dx \Big|_{s=0} = \int_{-\infty}^{\infty} x^n f_X(x) dx = \mathbf{E}[x^n]$$

the *n*-th moment of X

Illustrative Examples (1/2)

- **Example 4.27.** Given a random variable X with PMF:

$$p_X(x) = \begin{cases} 1/2, & \text{if } x = 2, \\ 1/6, & \text{if } x = 3, \\ 1/3, & \text{if } x = 5. \end{cases}$$

$$M_X(s) = \mathbf{E}\left[e^{sX}\right] = \sum_x e^{sx} p_X(x)$$
$$= \frac{1}{2}e^{2s} + \frac{1}{6}e^{3s} + \frac{1}{3}e^{5s}$$

$$\Rightarrow \mathbf{E}[X] = \frac{dM(s)}{ds}\Big|_{s=0}$$
$$= \frac{1}{2}\cdot 2\cdot e^{2s} + \frac{1}{6}\cdot 3\cdot e^{3s} + \frac{1}{3}\cdot 5\cdot e^{5s}\Big|_{s=0}$$
$$= 1 + \frac{3}{6} + \frac{5}{3} = \frac{19}{6}$$

$$\Rightarrow \mathbf{E}[X^2] = \frac{d^2M(s)}{d^2s}\Big|_{s=0}$$
$$= \frac{1}{2}\cdot 4\cdot e^{2s} + \frac{1}{6}\cdot 9\cdot e^{3s} + \frac{1}{3}\cdot 25\cdot e^{5s}\Big|_{s=0}$$
$$= 2 + \frac{9}{6} + \frac{25}{3} = \frac{71}{6}$$

Illustrative Examples (2/2)

- **Example.** Given an exponential random variable X with PMF:

$$f_X(x) = \lambda e^{-\lambda x}, \qquad x \geq 0.$$

$$M_X(s) = \int_0^\infty e^{sx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{(s-\lambda)x} dx$$

$$= \lambda \frac{e^{(s-\lambda)x}}{(s-\lambda)} \Big|_0^\infty \qquad (\text{if } s - \lambda < 0)$$

$$= \frac{\lambda}{\lambda - s}$$

$$\Rightarrow \mathbf{E}[X] = \frac{dM_X(s)}{ds} \Big|_{s=0} = \frac{\lambda}{(\lambda - s)^2} \Big|_{s=0} = \frac{1}{\lambda}$$

$$\Rightarrow \mathbf{E}[X^2] = \frac{d^2 M_X(s)}{d^2 s} \Big|_{s=0} = \frac{2\lambda}{(\lambda - s)^3} \Big|_{s=0} = \frac{2}{\lambda^2}$$

Two Properties of Transforms

- For any random variable X, we have

$$M_X(0) = \mathbf{E}\left[e^{0X}\right] = \mathbf{E}[1] = 1$$

- If random variable X only takes nonnegative integer values ($x = 0,1,2,\cdots$)

$$\lim_{s \to -\infty} M_X(s) = \mathbf{P}(X = 0)$$

$$\lim_{s \to -\infty} M_X(s) = \lim_{s \to -\infty} \sum_{k=0}^{\infty} \mathbf{P}(X = k) e^{sk} = \mathbf{P}(X = 0)$$

Inversion of Transforms

- Inversion Property
  - The transform $M_X(s)$ associated with a random variable X uniquely determines the probability law of X, assuming that $M_X(s)$ is finite for all s in an interval $[-a, a]$, $a \geq 0$
    - The determination of the probability law of a random variable => The PDF and CDF
    - In particular, if $M_X(s) = M_Y(s)$ for all s in $[-a, a]$, then the random variables X and Y have the same probability law

Illustrative Examples (1/2)

- **Example 4.28.** We are told that the transform associated with a random variable X is

$$M_X(s) = \frac{1}{4}e^{-s} + \frac{1}{2} + \frac{1}{8}e^{4s} + \frac{1}{8}e^{5s}$$

If we compare the formula $M_X(s) = \sum_x e^{sx} p_X(x)$, (if X is discrete)

we will have $p_X(-1) = \mathbf{P}(X = -1) = \frac{1}{4}$,

$$p_X(0) = \mathbf{P}(X = 0) = \frac{1}{2},$$

$$p_X(4) = \mathbf{P}(X = 4) = \frac{1}{8},$$

$$p_X(5) = \mathbf{P}(X = 5) = \frac{1}{8}.$$

Illustrative Examples (2/2)

- **Example 4.29. The Transform of a Geometric Random Variable.** We are told that the transform associated with random variable X is of the form

$$M_X(s) = \frac{pe^s}{1-(1-p)e^s}$$

  - Where $0 < p \leq 1$

If $(1-p)e^s < 1$, we can set $\alpha = (1-p)e^s$.

- Based on the property that

$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \ldots, \quad (\alpha < 1)$$

- $M_X(s)$ is then expressed as

$$M_X(s) = pe^s\left(1 + (1-p)e^s + (1-p)^2 e^{2s} + (1-p)^3 e^{3s} + \ldots\right)$$

- It can be infered that X is a discrete random variable with PDF

$$p_X(x) = p(1-p)^{x-1}, \quad x = 1,2,\ldots$$

$\therefore X$ is a geometric random variable

$$\mathbf{E}[X] = \frac{dM_X(s)}{ds}\Big|_{s=0}$$

$$= \frac{d\left(\frac{pe^s}{1-(1-p)e^s}\right)}{ds}\Big|_{s=0}$$

$$= \left[\frac{pe^s}{1-(1-p)e^s} + \frac{(1-p)pe^s}{\left(1-(1-p)e^s\right)^2}\right]\Big|_{s=0}$$

$$= 1 + \frac{(1-p)p}{p^2}$$

$$= \frac{1}{p}$$

Mixture of Distributions of Random Variables (1/3)

- Let $X_1, \ldots, X_n$ be continuous random variables with PDFs $f_{X_1}, \cdots, f_{X_n}$, and let Y be a random variable, which is equal to X_i with probability p_i ($\sum_{i=1}^{n} p_i = 1$). Then,

$$f_Y(y) = p_1 f_{X_1}(y) + \cdots + p_n f_{X_n}(y)$$

$$\text{(Note that this is quite different from : } Y = p_1 X_1 + \cdots + p_n X_n \text{)}$$

and

$$M_Y(s) = p_1 M_{X_1}(s) + \cdots + p_n M_{X_n}(s)$$

Mixture of Distributions of Random Variables (2/3)

$$f_Y(y) = p_1 f_{X_1}(y) + \cdots + p_n f_{X_n}(y), \quad \sum_{i=1}^{n} p_i = 1$$

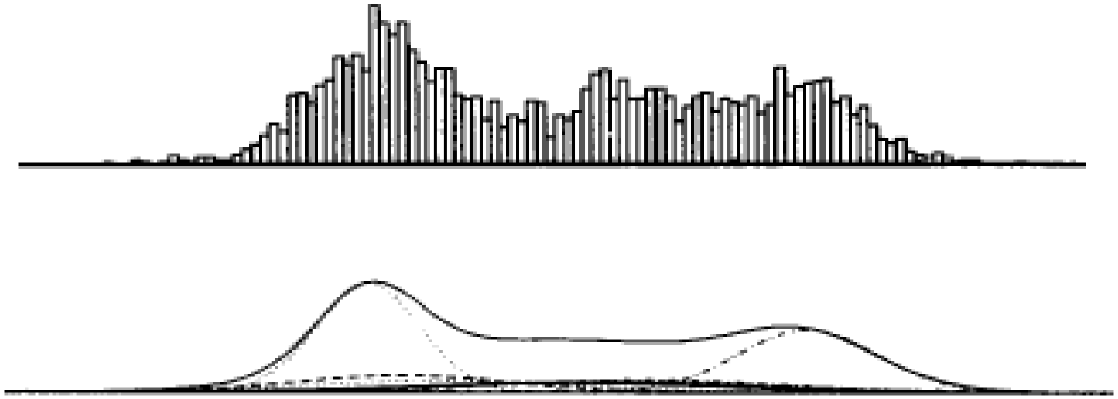
$$\begin{aligned}
M_Y(s) &= \mathbf{E}\left[e^{sY}\right] = \int_{-\infty}^{\infty} e^{sy} f_Y(y) dy \\
&= \int_{-\infty}^{\infty} e^{sy} \left(p_1 f_{X_1}(y) + \cdots + p_n f_{X_n}(y)\right) dy \\
&= \left[\int_{-\infty}^{\infty} e^{sy} p_1 f_{X_1}(y) dy\right] + \cdots + \left[\int_{-\infty}^{\infty} e^{sy} p_n f_{X_n}(y) dy\right] \\
&= \left[p_1 \int_{-\infty}^{\infty} e^{sx_1} f_{X_1}(x_1) dx_1\right] + \cdots + \left[p_n \int_{-\infty}^{\infty} e^{sx_n} f_{X_n}(x_n) dx_n\right] \\
&= p_1 M_{X_1}(s) + \cdots + p_n M_{X_n}(s)
\end{aligned}$$

Mixture of Distributions of Random Variables (3/3)

- **Mixture of Gaussian Distributions**
  - More complex distributions with multiple local maxima can be approximated by Gaussian (a unimodal distribution) mixture

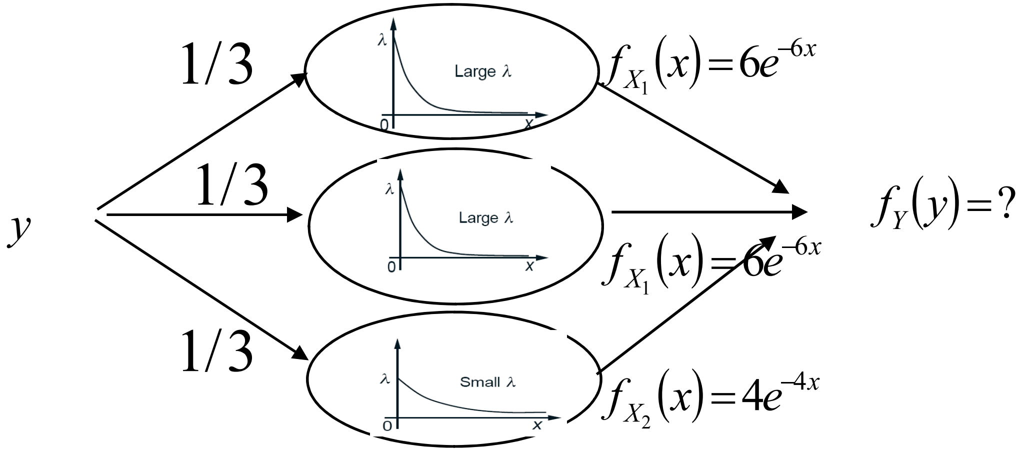
$$f_Y(y)=\sum_{i=1}^{n} p_i N_i\left(y;\mu_i,\sigma_i^2\right), \qquad \sum_{i=1}^{n} p_i = 1$$

  - Gaussian mixtures with enough mixture components can approximate any distribution



An Illustrative Example (1/2)

- **Example 4.30. The Transform of a Mixture of Two Distributions.** The neighborhood bank has three tellers, two of them fast, one slow. The time to assist a customer is exponentially distributed with parameter *λ* = 6 at the fast tellers, and *λ* = 4 at the slow teller. Jane enters the bank and chooses a teller at random, each one with probability 1/3. Find the PDF of the time it takes to assist Jane and the associated transform



y

$1/3$

$1/3$

$1/3$

$f_{X_1}(x) = 6e^{-6x}$

$f_{X_1}(x) = 6e^{-6x}$

$f_{X_2}(x) = 4e^{-4x}$

$f_Y(y) = ?$

An Illustrative Example (2/2)

- The service time of each teller is exponentially distributed

$$f_{X_1}(x)=6e^{-6x}, \qquad x\geq 0.$$

the faster teller

$$f_{X_2}(x)=4e^{-4x}, \qquad x\geq 0.$$

the slower teller

- The distribution of the time that a customer spends in the bank

$$f_Y(y)=\frac{2}{3}\cdot 6e^{-6y}+\frac{1}{3}\cdot 4e^{-4y}, \qquad y\geq 0.$$

  - The associated transform

$$M_Y(s)=\mathbf{E}\left[e^{sy}\right]=\int_0^{\infty}e^{sy}\left(\frac{2}{3}\cdot 6e^{-6y}+\frac{1}{3}\cdot 4e^{-4y}\right)dy$$

$$=\frac{2}{3}\int_0^{\infty}e^{sy}\cdot 6e^{-6y}dy+\frac{1}{3}\int_0^{\infty}e^{sy}\cdot 4e^{-4y}dy$$

$$=\frac{2}{3}\cdot\frac{6}{6-s}+\frac{1}{3}\cdot\frac{4}{4-s} \qquad (\text{for } s<4)$$

Sum of Independent Random Variables

- Addition of **independent** random variables corresponds to multiplication of their transforms
  - Let X and Y be independent random variables, and let $W = X + Y$. The transform associated with W is,

$$M_W(s) = \mathbf{E}\left[e^{sW}\right] = \mathbf{E}\left[e^{s(X+Y)}\right] = \mathbf{E}\left[e^{sX} e^{sY}\right] = \mathbf{E}\left[e^{sX}\right]\mathbf{E}\left[e^{sY}\right] = M_X(s) M_Y(s)$$

    - Since X and Y are independent, and e^{sX} and e^{sY} are functions of X and Y , respectively
- More generally, if $X_1, \ldots, X_n$ is a collection of independent random variables, and $W = X_1 + \cdots + X_n$

$$M_W(s) = M_{X_1}(s) \cdots M_{X_n}(s)$$

Illustrative Examples (1/3)

- **Example 4.10. The Transform of the Binomial.**

  Let $X_1, \ldots, X_n$ be independent Bernoulli random variables with a common parameter p . Then,

$$M_{X_i}(s) = (1-p)e^{s \cdot 0} + pe^{s \cdot 1} = 1 - p + pe^s, \quad \text{for } i = 1, \ldots, n$$

  - If $Y = X_1 + \cdots + X_n$, Y can be thought of as a binomial random variable with parameters n and p , and its corresponding transform is given by

$$M_Y(s) = \prod_{i=1}^{n} M_{X_i}(s) = \left(1 - p + pe^s\right)^n$$

Illustrative Examples (2/3)

- **Example 4.11. The Sum of Independent Poisson Random Variables is Poisson.**
  - Let X and Y be independent Poisson random variables with means λ and μ , respectively
    - The transforms of X and Y will be the following, respectively

$$M_X(s) = e^{\lambda(e^s - 1)}, \; M_Y(s) = e^{\mu(e^s - 1)}$$

cf. p.5 (in this lecture)

  - If $W = X + Y$, then the transform of the random variable W is

$$\begin{aligned} M_W(s) &= M_X(s) M_Y(s) \\ &= e^{\lambda(e^s - 1)} e^{\mu(e^s - 1)} \\ &= e^{(\lambda + \mu)(e^s - 1)} \end{aligned}$$

    - From the transform of W , we can conclude that W is also a Poisson random variable with mean $\lambda + \mu$

Illustrative Examples (3/3)

- **Example 4.12. The Sum of Independent Normal Random Variables is Normal.**
  - Let X and Y be independent normal random variables with means μ_x, μ_y , and variances σ_x^2, σ_y^2 , respectively
    - The transforms of X and Y will be the following, respectively

$$M_X(s) = e^{\frac{\sigma_x^2 s^2}{2} + \mu_x s}, \; M_Y(s) = e^{\frac{\sigma_y^2 s^2}{2} + \mu_y s}$$

cf. p.8 (in this lecture)

  - If $W = X + Y$, then the transform of the random variable W is

$$M_W(s) = M_X(s) M_Y(s)$$

$$= e^{\frac{\left(\sigma_x^2 + \sigma_y^2\right)s^2}{2} + \left(\mu_x + \mu_y\right)s}$$

    - From the transform of W , we can conclude that W also is normal with mean $\mu_x + \mu_y$ and variance $\sigma_x^2 + \sigma_y^2$

Tables of Transforms (1/2)

Transforms for Common Discrete Random Variables

Bernoulli(p)

$$p_X(k) = \begin{cases} p, & \text{if } k = 1, \\ 1 - p, & \text{if } k = 0. \end{cases} \qquad M_X(s) = 1 - p + pe^s.$$

Binomial(n, p)

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n.$$

$$M_X(s) = (1 - p + pe^s)^n.$$

Geometric(p)

$$p_X(k) = p(1-p)^{k-1}, \quad k = 1, 2, \ldots \qquad M_X(s) = \frac{pe^s}{1 - (1-p)e^s}.$$

Poisson(λ)

$$p_X(k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, \ldots \qquad M_X(s) = e^{\lambda(e^s - 1)}.$$

Uniform(a, b)

$$p_X(k) = \frac{1}{b - a + 1}, \quad k = a, a+1, \ldots, b.$$

$$M_X(s) = \frac{e^{as}}{b - a + 1} \frac{e^{(b-a+1)s} - 1}{e^s - 1}.$$

Tables of Transforms (2/2)

Transforms for Common Continuous Random Variables

Uniform(a, b)

$$f_X(x) = \frac{1}{b-a}, \quad a \le x \le b. \qquad M_X(s) = \frac{1}{b-a}\frac{e^{sb} - e^{sa}}{s}.$$

Exponential(λ)

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \ge 0. \qquad M_X(s) = \frac{\lambda}{\lambda - s}, \qquad (s > \lambda).$$

Normal(μ, σ^2)

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty. \qquad M_X(s) = e^{\frac{\sigma^2 s^2}{2} + \mu s}.$$

Exercise

- Given that X is an exponential random variable with parameter λ :

  (i) Show that the transform (moment generating function) of can be expressed as:

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

  (ii) Find the expectation and variance of X based on its transform.

  (ii) Given that random variable Y can be expressed as $Y = 3X + 5$. Find the transform of Y .

  (iv) Given that Z is also an exponential random variable with parameter η , and X and Z are independent. Find the transform of random variable $W = 3X + 2Z$.

Sum of a Random Number of Independent Random Variables (1/4)

$$\underbrace{X_1, X_2, X_3, \ldots, X_N}_{Y = X_1 + X_2 + \cdots + X_N}, \ldots \ldots \ldots$$

- If we know that
  - N is a random variable taking positive integer values $N = 1, 2, \ldots$
  - $X_1, X_2, \ldots$ are independent, identically distributed (i.i.d.) random variables (with common mean μ and variance σ^2)
    - A subset of $X_i's$ ($X_1, X_2, \cdots, X_N$) are independent as well

- What are the formulas for the mean, variance, and the transform of Y ? (If $N = 0$, we let $Y = 0$)

$$Y = X_1 + X_2 + \cdots + X_N$$

Sum of a Random Number of Independent Random Variables (2/4)

- If we fix some number n, the random variable $X_1 + X_2 + \cdots + X_n$ is independent of random variable N

$$
\begin{aligned}
&\mathbf{E}\left[Y \middle| N = n\right] \\
&= \mathbf{E}\left[X_1 + X_2 + \cdots + X_N \middle| N = n\right] \\
&= \mathbf{E}\left[X_1 + X_2 + \cdots + X_n \middle| N = n\right] \\
&= \mathbf{E}\left[X_1 + X_2 + \cdots + X_n\right] \\
&= n\mathbf{E}\left[X_i\right] = n\mu
\end{aligned}
$$

?

  - $\mathbf{E}\left[Y \middle| N\right]$ can be viewed as a function of random variable N
    - $\mathbf{E}\left[Y \middle| N\right]$ is a random variable
    - The mean of $\mathbf{E}\left[Y \middle| N\right]$ (i.e. $\mathbf{E}\left[Y\right]$) can be calculated by using the law of iterated expectations

$$
\mathbf{E}\left[Y\right] = \mathbf{E}\left[\mathbf{E}\left[Y \middle| N\right]\right] = \mathbf{E}\left[N\mu\right] = \mu\mathbf{E}\left[N\right]
$$

Sum of a Random Number of Independent Random Variables (3/4)

- Similarly, $\operatorname{var}(Y \mid N = n)$ can be expressed as

$$\begin{aligned}
&\operatorname{var}(Y \mid N = n) \\
&= \operatorname{var}(X_1 + X_2 + \cdots + X_N \mid N = n) \\
&= \operatorname{var}(X_1 + X_2 + \cdots + X_n \mid N = n) \\
&= \operatorname{var}(X_1 + X_2 + \cdots + X_n) \\
&= n\sigma^2
\end{aligned}$$

  - $\operatorname{var}(Y \mid N)$ can be viewed as a function of random variable N
    - $\operatorname{var}(Y \mid N)$ is a random variable
  - The variance of Y can be calculated using the law of total variance

$$\begin{aligned}
\operatorname{var}(Y) &= \mathbf{E}[\operatorname{var}(Y \mid N)] + \operatorname{var}(\mathbf{E}[Y \mid N]) \\
&= \mathbf{E}[N\sigma^2] + \operatorname{var}(N\mu) \\
&= \sigma^2 \mathbf{E}[N] + \mu^2 \operatorname{var}(N)
\end{aligned}$$

Sum of a Random Number of Independent Random Variables (4/4)

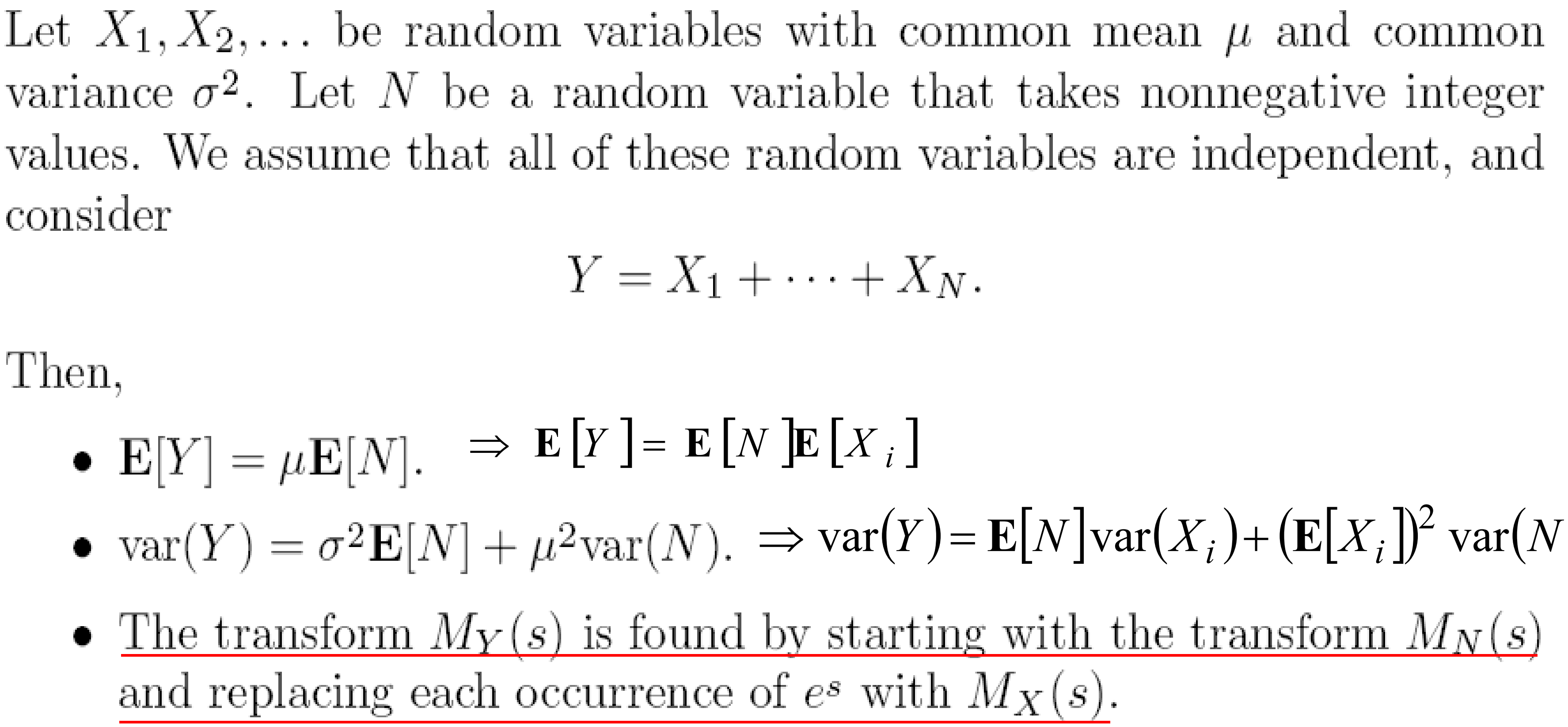
- Similarly, $\mathbf{E}\left[e^{sY} \mid N = n\right]$ can be expressed as

$$\begin{aligned}
&\mathbf{E}\left[e^{sY} \mid N = n\right] \\
&= \mathbf{E}\left[e^{s(X_1+X_2+\cdots+X_N)} \middle| N = n\right] = \mathbf{E}\left[e^{s(X_1+X_2+\cdots+X_n)} \middle| N = n\right] \\
&= \mathbf{E}\left[e^{s(X_1+X_2+\cdots+X_n)}\right] = \mathbf{E}\left[e^{sX_1} e^{sX_2} \cdots e^{sX_n}\right] \\
&= \left(M_X(s)\right)^n
\end{aligned}$$

  - $\mathbf{E}\left[e^{sY} \mid N\right]$ can be viewed as a function of random variable N
    - $\mathbf{E}\left[e^{sY} \mid N\right]$ is a random variable
    - The mean of $\mathbf{E}\left[e^{sY} \mid N\right]$ (i.e. the transform of Y , $\mathbf{E}\left[e^{sY}\right]$) can be calculated by using the law of iterated expectations

$$M_Y(s) = \mathbf{E}\left[e^{sY}\right] = \mathbf{E}\left[\mathbf{E}\left[e^{sY} \mid N\right]\right] = \mathbf{E}\left[\left(M_X(s)\right)^N\right] = \sum_{n=1}^{\infty} \left(M_X(s)\right)^n p_N(n)$$

Properties of the Sum of a Random Number of Independent Random Variables



Let $X_1, X_2, \ldots$ be random variables with common mean μ and common variance σ^2. Let N be a random variable that takes nonnegative integer values. We assume that all of these random variables are independent, and consider

$$Y = X_1 + \cdots + X_N.$$

Then,

- $\mathbf{E}[Y] = \mu\mathbf{E}[N].$ $\Rightarrow \mathbf{E}[Y] = \mathbf{E}[N]\mathbf{E}[X_i]$
- $\text{var}(Y) = \sigma^2\mathbf{E}[N] + \mu^2\text{var}(N).$ $\Rightarrow \text{var}(Y) = \mathbf{E}[N]\text{var}(X_i) + (\mathbf{E}[X_i])^2 \text{var}(N)$
- The transform $M_Y(s)$ is found by starting with the transform $M_N(s)$ and replacing each occurrence of e^s with $M_X(s)$.

Illustrative Examples (1/5)

- **Example 4.34.** A remote village has three gas stations, and each one of them is open on any given day with probability 1/2, independently of the others. The amount of gas available in each gas station is unknown and is uniformly distributed between 0 and 1000 gallons.
  - We wish to characterize the distribution (Y) of the total amount of gas available at the gas stations that are open

$$Y = X_1 + \cdots$$

Total amount of gas available

The amount of gas provided by one gas station, out of three (X_i is uniformly distributed)

①

The number N of gas stations open at a day is a binomial distributi on with parameter $(3, p)$

$\Rightarrow$ the transform of random variable N is

$$M_N(s) = \left(1 - p + pe^s\right)^3 = \frac{1}{8}\left(1 + e^s\right)^3$$

② The transform of $\mathrm{X_i}$ (uniformlly distribute d) is :

$$M_X(s) = \int_0^{1000} e^{sx} \cdot \frac{1}{1000} dx$$

$$= \frac{e^{1000s} - 1}{1000s}$$

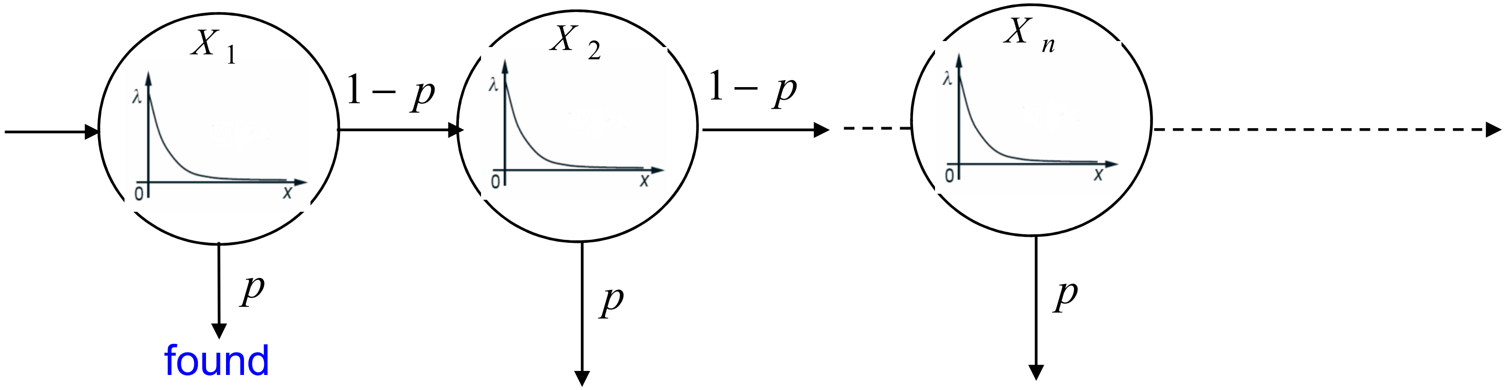
③ Using the property introduced in the previous slide, we have

$$M_Y(s) = \frac{1}{8}\left(1 + \left(\frac{e^{1000\,s} - 1}{1000\,s}\right)\right)^3$$

Illustrative Examples (2/5)

- **Example 4.35. Sum of a Geometric Number of Independent Exponential Random Variables.**
  - Jane visits a number of bookstores, looking for ***Great Expectations***. Any given bookstore carries the book with probability p, independently of the others. In a typical bookstore visited, Jane spends a random amount of time, exponentially distributed with parameter λ, until she either finds the book or she decides that the bookstore does not carry it. Assuming that Jane will keep visiting bookstores until she buys the book and that the time spent in each is independent of everything else
  - We wish to determine the mean, variance, and PDF of the total time spent in bookstores.

Illustrative Examples (3/5)



found

$Y = X_1$

found

$Y = X_1 + X_2$

found

$Y = X_1 + X_2 + \cdots + X_n$

Total amount of time spent

The amount of time spent in a given bookstore

①

$$\Rightarrow \mathbf{E}[Y] = \mathbf{E}[N]\mathbf{E}[X_i] = \frac{1}{p}\cdot\frac{1}{\lambda}$$

(The mean of exponentia l distributi on with parameter λ is $\frac{1}{\lambda} \Rightarrow \mathbf{E}[X_i] = \frac{1}{\lambda}$

The mean geometric distributi on with parameter p is $\frac{1}{p} \Rightarrow \mathbf{E}[N] = \frac{1}{p}$)

②

$$\mathrm{var}(Y) = \mathbf{E}[N]\mathrm{var}(X_i) + (\mathbf{E}[X_i])^2\,\mathrm{var}(N)$$

$$= \frac{1}{p}\cdot\frac{1}{\lambda^2} + \left(\frac{1}{\lambda}\right)^2 \cdot \frac{1-p}{p^2}$$

$$= \frac{1}{\lambda^2 p^2}$$

Illustrative Examples (4/5)

③

$$M_X(s) = \frac{\lambda}{\lambda - s}$$

$$M_N(s) = \frac{pe^s}{1-(1-p)e^s}$$

$$\Rightarrow M_Y(s) = \frac{p\dfrac{\lambda}{\lambda - s}}{1-(1-p)\dfrac{\lambda}{\lambda - s}} = \frac{p\lambda}{\lambda - s - \lambda + p\lambda} = \frac{p\lambda}{p\lambda - s}$$

$\therefore$ Y is an exponentia lly distribute d random variable with parameter $p\lambda$

$$f_Y(y) = p\lambda e^{-p\lambda y}, \quad y \geq 0$$

Recall that if Y is the sum of a fixed number of independent random variables (e.g., $Y = X_1 + X_2$), its associated transform $M_Y(s)$ is (Assume that X_1, X_2 are identical exponential distributions with parameter λ)

$$M_Y(s) = \left(\frac{\lambda}{\lambda - s}\right)^2$$

$\Rightarrow$ Y is not an exponential random variable

Illustrative Examples (5/5)

- **Example 4.36. Sum of a Geometric Number of Independent Geometric Random Variables.**
  - This example is a discrete counterpart of the preceding one.
  - We let N be geometrically distributed with parameter p. We also let each random variable X_i be geometrically distributed with parameter q. We assume that all of these random variables are independent.

$$M_X(s) = \frac{qe^s}{1-(1-q)e^s}$$

$$M_N(s) = \frac{pe^s}{1-(1-p)e^s}$$

$$\Rightarrow M_Y(s) = \frac{p\dfrac{qe^s}{1-(1-q)e^s}}{1-(1-p)\dfrac{qe^s}{1-(1-q)e^s}} = \frac{pqe^s}{1-(1-q)e^s-(1-p)qe^s} = \frac{pqe^s}{1-(1-pq)e^s}$$

$$\therefore \ Y \text{ is a geometric distributed random variable with parameter } pq$$

Exercise

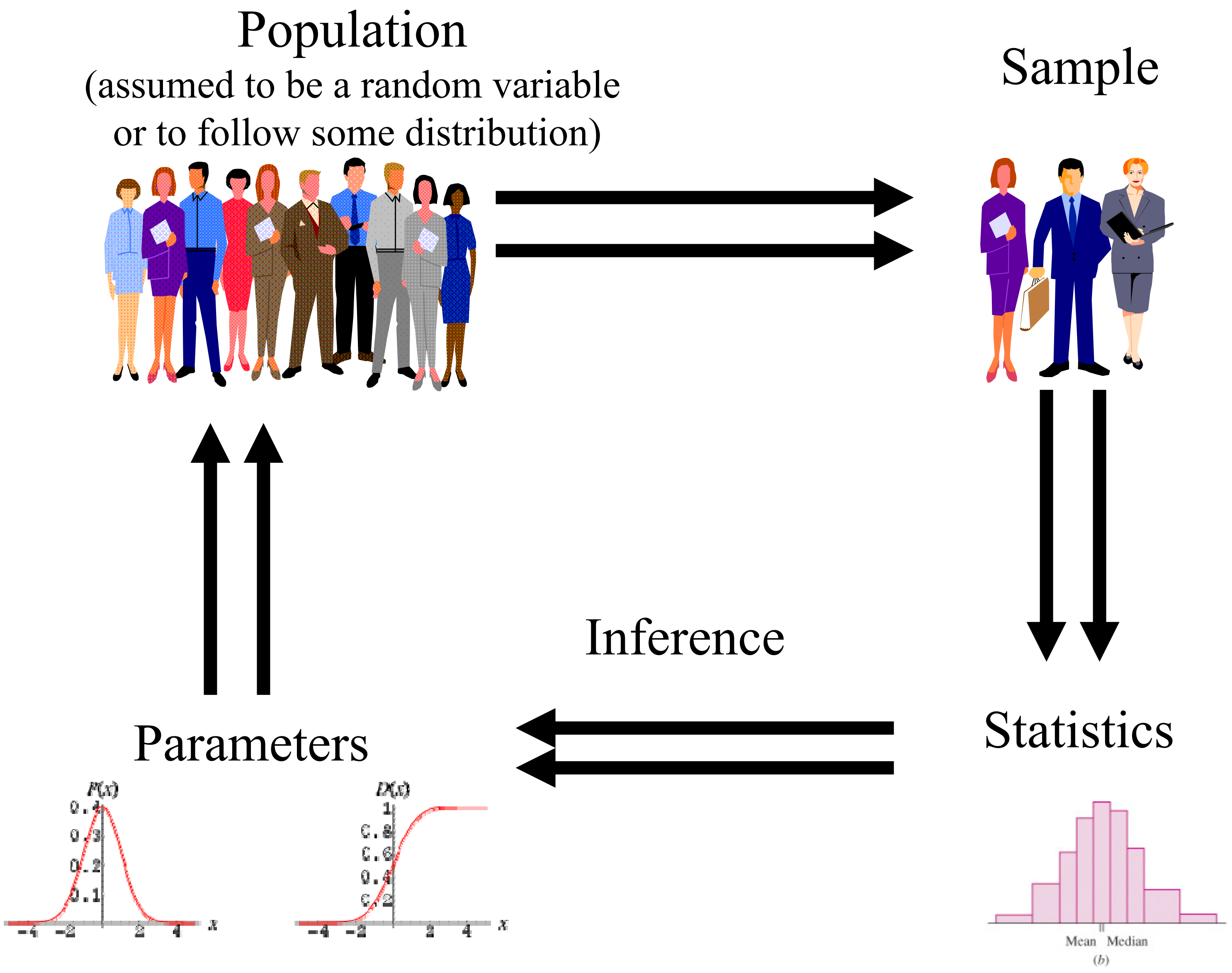
- A fair coin is flipped independently until the first head is encountered. For each time of the coin flipping, you will get a score of 1 with probability 0.4 and a score of 0 with probability 0.6. Let the random variable Y be defined as the sum of all the scores obtained during the process of the coin flipping (including the last time of the coin flipping). Find the following characteristics of Y:

  (i) mean

  (ii) variance

  (iii) transform

Probability versus Statistics



Population
(assumed to be a random variable
or to follow some distribution)

Sample

Statistics

Inference

Parameters

The Central Limit Theorem (1/2)

- The Central Limit Theorem
  - Let $X_1, \ldots, X_n$ be a sequence of independent, identically distributed random variables with common mean μ and variance σ^2
  - Let $X' = \dfrac{X_1 + \cdots + X_n}{n}$ be an (sample) average of these random variables
  - Let $S_n = X_1 + \ldots + X_n$ be the sum of these random variables

  Then if n is sufficiently large:

    - $X' \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$ X' is a normal with mean μ and variance $\dfrac{\sigma^2}{n}$
    - And $S_n \sim N(n\mu, n\sigma^2)$ approximately

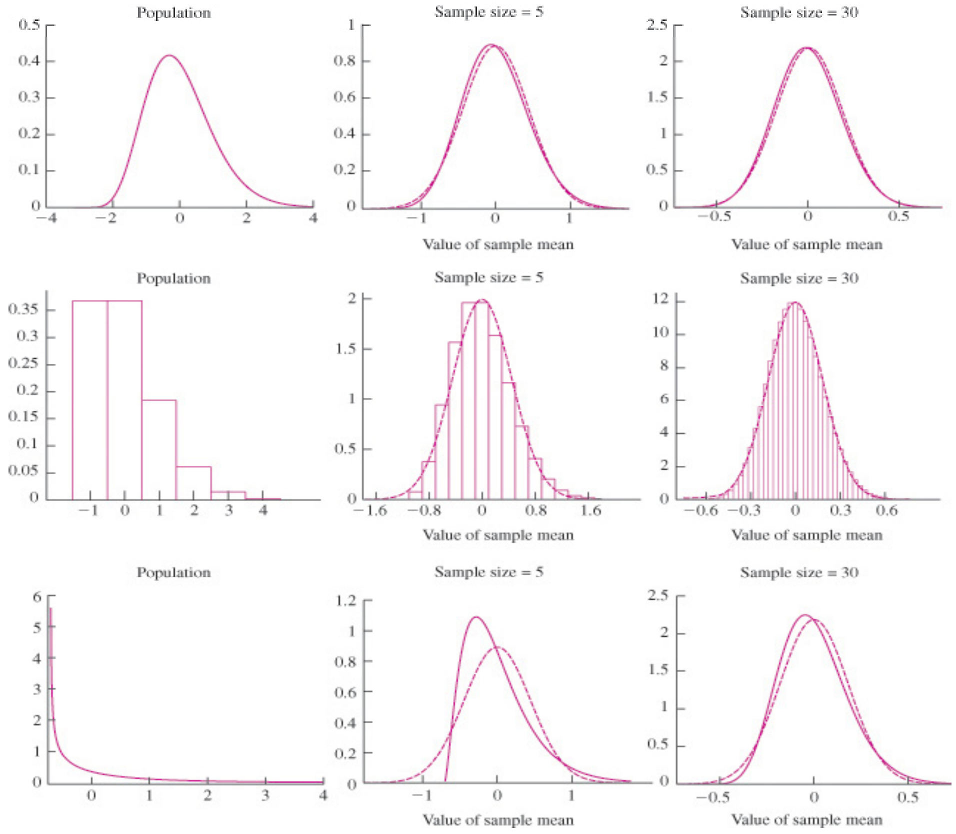
The Central Limit Theorem (2/2)

- Example

continuous

discrete

continuous



Solid lines: real distributions of sample mean

Dashed lines: normal distributions

somewhat skewed to the right

highly skewed to the right

- Rule of Thumb
  - For most populations, if the (sample) size *n* is greater than 30, the Central Limit Theorem approximation is good

Chebyshev Inequality

- If X is a random variable with mean μ and variance σ^2 , then

$$\mathbf{P}\left(|X-\mu| \geq c\right) \leq \frac{\sigma^2}{c^2} \qquad \text{for all } c > 0$$

or alternatively,

$$\mathbf{P}\left(|X-\mu| \geq k\sigma\right) \leq \frac{1}{k^2} \qquad \text{for all } k > 0$$

Proof : (assume here that X is a continuous random variable)

We introduce a function of X

$$g(x) = \begin{cases} 0, & \text{if } |x\text{-}\mu| < c \\ c^2, & \text{if } |x\text{-}\mu| \geq c \end{cases} \quad \left(\text{Note that } (x\text{-}\mu)^2 \overset{?}{\geq} g(x) \text{ for all } x\right)$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x\text{-}\mu)^2 f(x)dx \geq \int_{-\infty}^{\infty} g(x)f(x)dx = c^2\mathbf{P}\left(|x\text{-}\mu| \geq c\right)$$

$$\therefore\ \mathbf{P}\left(|x\text{-}\mu| \geq c\right) \leq \frac{\sigma^2}{c^2}$$