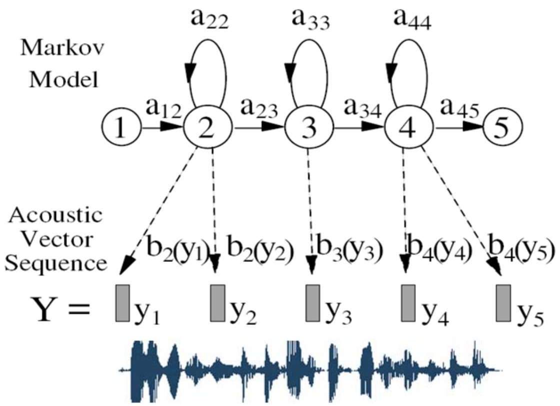

Hidden Markov Models for Speech Recognition

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References:

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2. Huang et. al. *Spoken Language Processing*. Chapters 4, 8
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Hidden Markov Model (HMM): A Brief Overview

History

- Published in papers of Baum in late 1960s and early 1970s
- Introduced to speech processing by Baker (CMU) and Jelinek (IBM) in the 1970s (discrete HMMs)
- Then extended to continuous HMMs by Bell Labs

Assumptions

- Speech signal can be characterized as a parametric random (stochastic) process
- Parameters can be estimated in a precise, well-defined manner

Three fundamental problems

- Evaluation of probability (likelihood) of a sequence of observations given a specific HMM
- Determination of a best sequence of model states
- Adjustment of model parameters so as to best account for observed signals (or discrimination purposes)

Stochastic Process

- A stochastic process is a mathematical model of a probabilistic experiment that evolves in time and generates a sequence of numeric values
  - Each numeric value in the sequence is modeled by a random variable
  - A stochastic process is just a (finite/infinite) sequence of random variables
- Examples
  - (a) The sequence of recorded values of a speech utterance
  - (b) The sequence of daily prices of a stock
  - (c) The sequence of hourly traffic loads at a node of a communication network
  - (d) The sequence of radar measurements of the position of an airplane

Observable Markov Model

- Observable Markov Model (Markov Chain)
  - **First-order** Markov chain of N states is a triple $(\boldsymbol{S},\boldsymbol{A},\pi)$
    - $\boldsymbol{S}$ is a set of N states
    - $\boldsymbol{A}$ is the $N \times N$ matrix of transition probabilities between states
      $P(s_t=j|s_{t-1}=i, s_{t-2}=k, \ldots\ldots) \approx P(s_t=j|s_{t-1}=i) \approx A_{ij}$ First-order and time-invariant assumptions
    - π is the vector of initial state probabilities
      $\pi_j = P(s_1=j)$
  - The output of the process is the set of states at each instant of time, when each state corresponds to an observable event
  - The output in any given state is not random (***deterministic!***)
  - Too simple to describe the speech signal characteristics

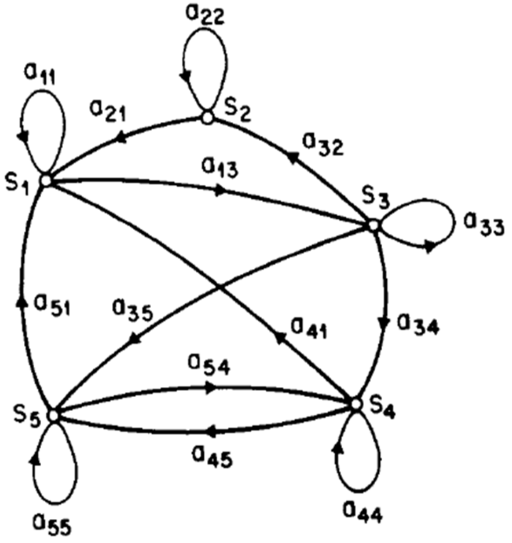
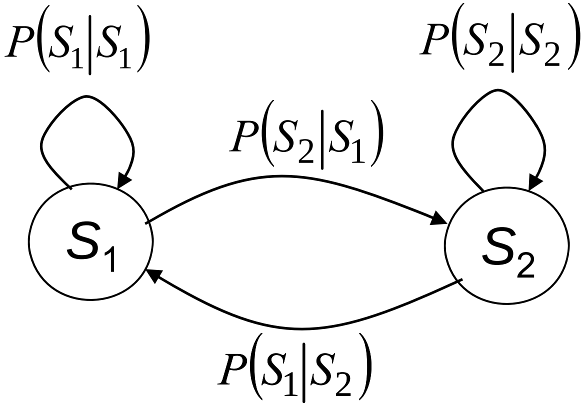
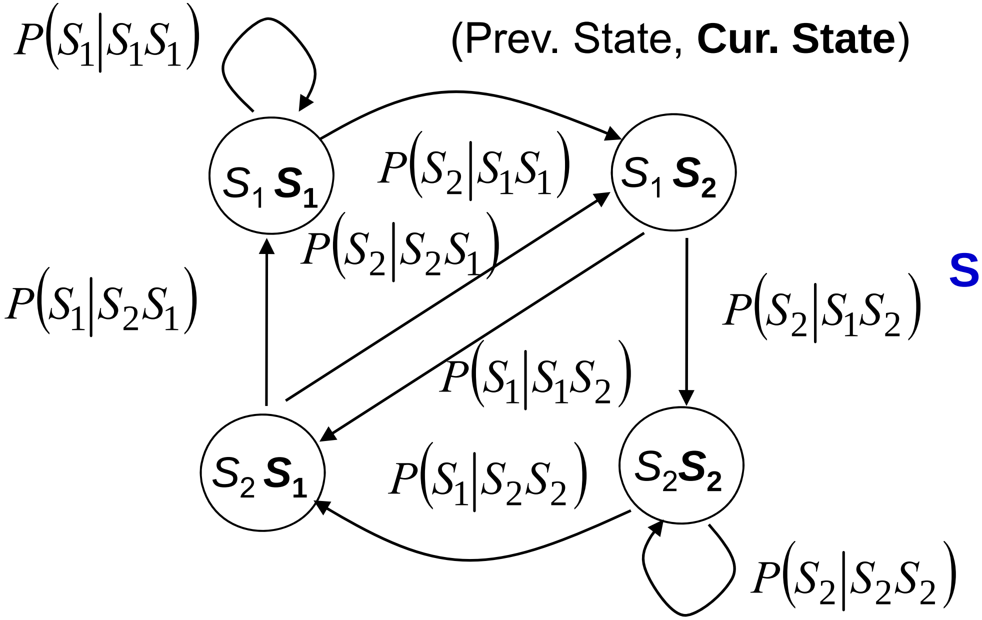


Fig. 1. A Markov chain with 5 states (labeled S_1 to S_5) with selected state transitions.

Observable Markov Model (cont.)



First-order Markov chain of *2* states



Second-order Markov chain of *2* states

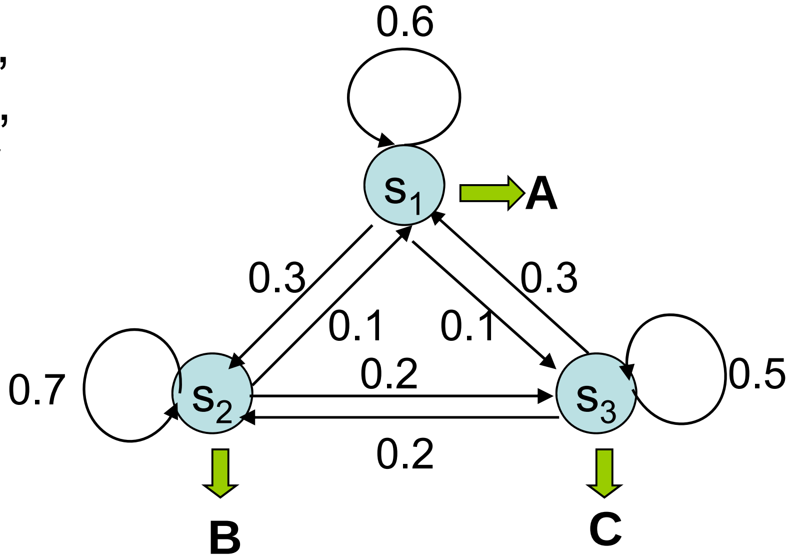
Observable Markov Model (cont.)

- Example 1: A 3-state Markov Chain λ

  State 1 generates symbol A **only**,
  State 2 generates symbol B **only**,
  and State 3 generates symbol C **only**

$$\mathbf{A} = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

$$\pi = \begin{bmatrix} 0.4 & 0.5 & 0.1 \end{bmatrix}$$

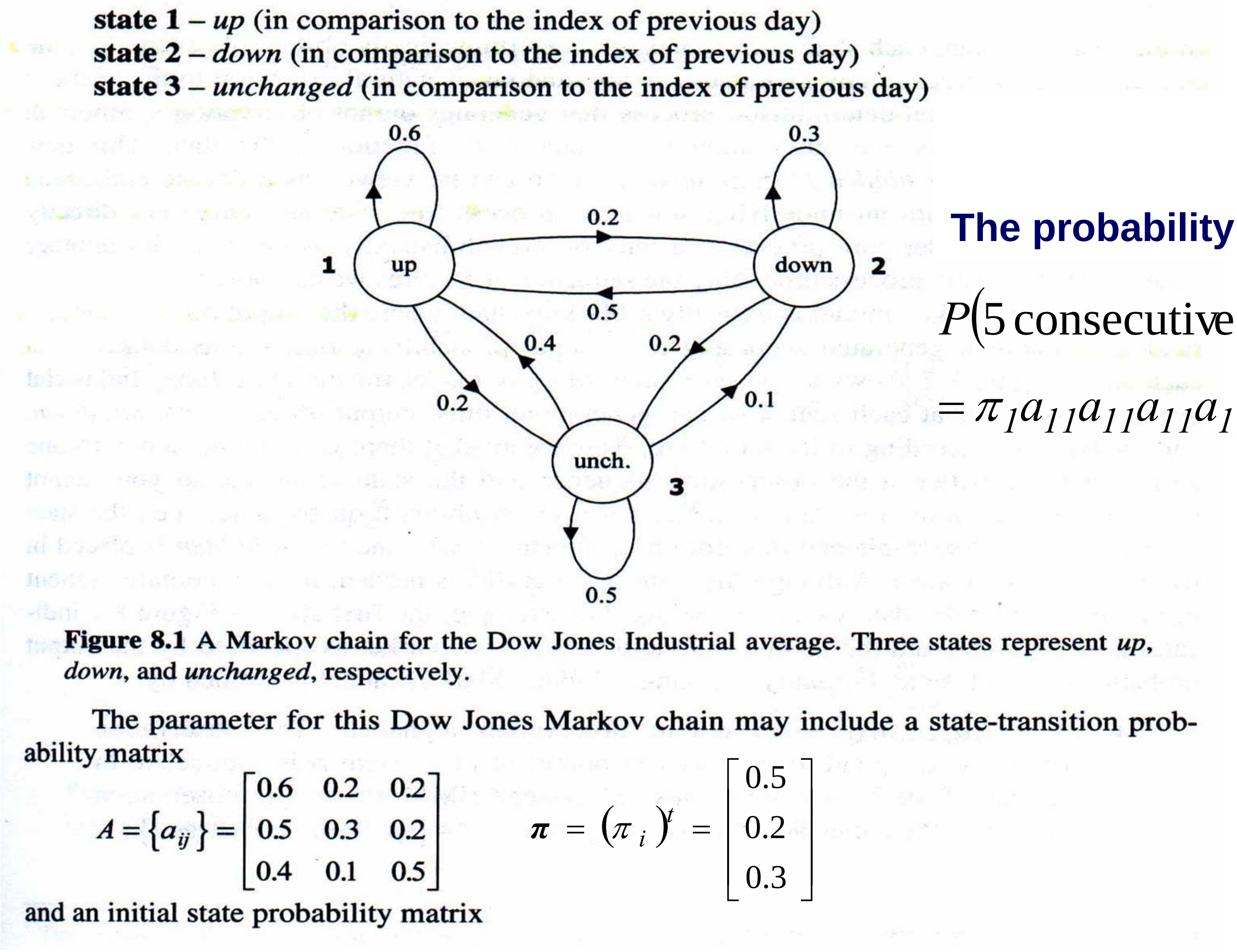


  - Given a sequence of observed symbols ***O***={CABBCABC}, the **only one** corresponding state sequence is {$S_3S_1S_2S_2S_3S_1S_2S_3$}, and the corresponding probability is

    $P(\mathbf{O}|\lambda)$
    $=P(S_3)P(S_1|S_3)P(S_2|S_1)P(S_2|S_2)P(S_3|S_2)P(S_1|S_3)P(S_2|S_1)P(S_3|S_2)$
    $=0.1\times0.3\times0.3\times0.7\times0.2\times0.3\times0.3\times0.2=0.00002268$

Observable Markov Model (cont.)

- Example 2: A three-state Markov chain for the *Dow Jones Industrial average*



state 1 – *up* (in comparison to the index of previous day)
state 2 – *down* (in comparison to the index of previous day)
state 3 – *unchanged* (in comparison to the index of previous day)

Figure 8.1 A Markov chain for the Dow Jones Industrial average. Three states represent *up*, *down*, and *unchanged*, respectively.

The parameter for this Dow Jones Markov chain may include a state-transition probability matrix

$$A = \{a_{ij}\} = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.5 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.5 \end{bmatrix} \qquad \boldsymbol{\pi} = (\pi_i)^t = \begin{bmatrix} 0.5 \\ 0.2 \\ 0.3 \end{bmatrix}$$

and an initial state probability matrix

The probability of 5 consecutive *up* days

$$P(5\,\text{consecutive}\,up\,\text{days}) = P(1,1,1,1,1)$$
$$= \pi_1 a_{11} a_{11} a_{11} a_{11} = 0.5 \times (0.6)^4 = 0.0648$$

Observable Markov Model (cont.)

- Example 3: Given a Markov model, what is the mean occupancy duration of each state *i*

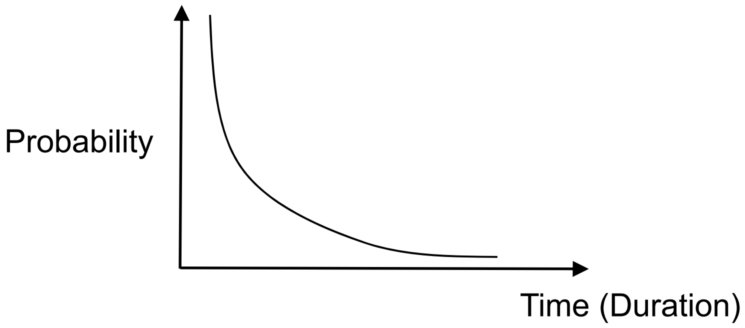
$$P_i(d) = \text{probability mass function of duration } d \text{ in state } i$$

$$= (a_{ii})^{d-1}(1-a_{ii}) \qquad \text{a geometric distribution}$$

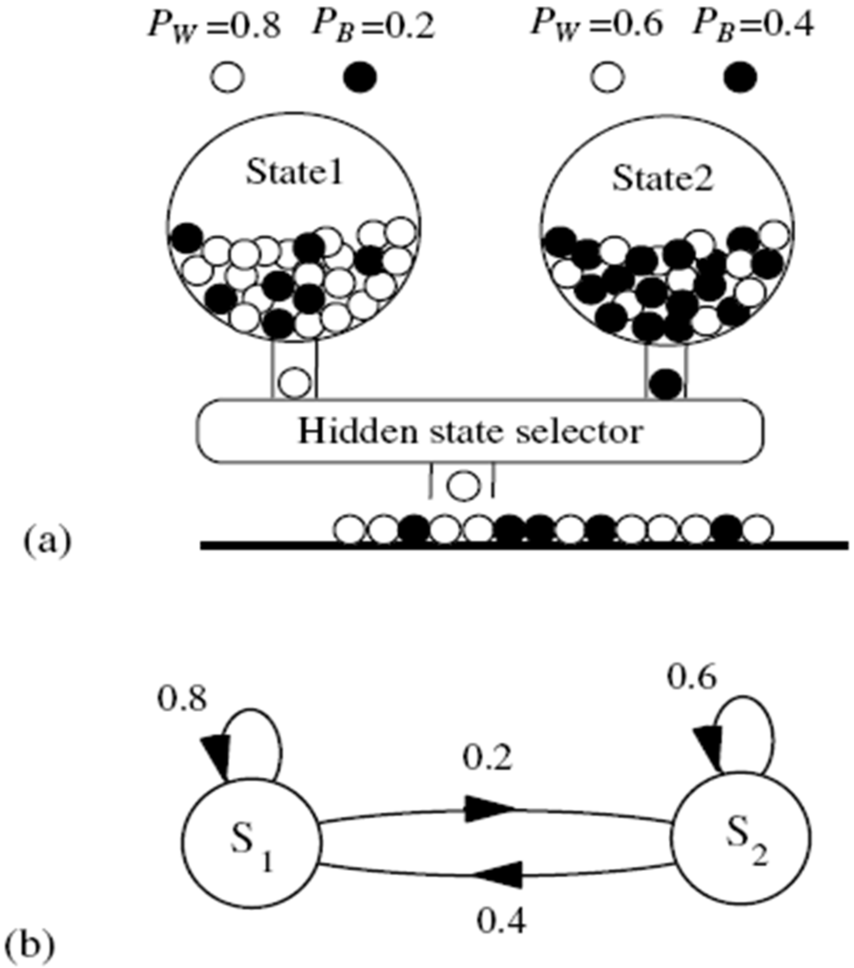
Expected number of duration in a state

$$\bar{d}_i = \sum_{d=1}^{\infty} dP_i(d) = \sum_{d=1}^{\infty} d(a_{ii})^{d-1}(1-a_{ii}) = (1-a_{ii})\frac{\partial}{\partial a_{ii}}\sum_{d=1}^{\infty}(a_{ii})^d$$

$$= (1-a_{ii})\frac{\partial}{\partial a_{ii}}\frac{1}{1-a_{ii}} = \frac{1}{1-a_{ii}}$$



Hidden Markov Model



(a) Illustration of a two-layered random process. (b) An HMM model of the process in (a).

Hidden Markov Model (cont.)

- HMM, an extended version of Observable Markov Model
  - The observation is turned to be a probabilistic function (discrete or continuous) of a state instead of an one-to-one correspondence of a state
  - The model is a doubly embedded stochastic process with an underlying stochastic process that is not directly observable (hidden)
    - What is hidden? ***The State Sequence!***
      According to the observation sequence, we are not sure which state sequence generates it!

- Elements of an HMM (the State-Output HMM) $\lambda=\{\boldsymbol{S},\boldsymbol{A},\boldsymbol{B},\pi\}$
  - $\boldsymbol{S}$ is a set of N states
  - $\boldsymbol{A}$ is the $N \times N$ matrix of transition probabilities between states
  - $\boldsymbol{B}$ is a set of N probability functions, each describing the observation probability with respect to a state
  - π is the vector of initial state probabilities

Hidden Markov Model (cont.)

- Two major assumptions
  - First order (Markov) assumption
    - The state transition depends only on the origin and destination
    - Time-invariant

$$P(s_t = j | s_{t-1} = i) = P(s_\tau = j | s_{\tau-1} = i) = P(j|i) = A_{i,j}$$

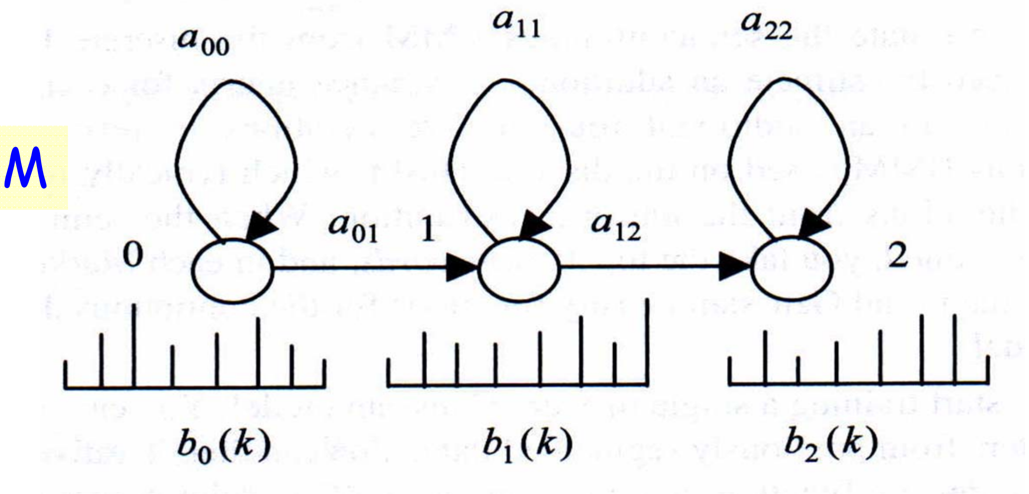
  - Output-independent assumption
    - All observations are dependent on the state that generated them, not on neighboring observations

$$P(\mathbf{o}_t | s_t, \ldots, \mathbf{o}_{t-2}, \mathbf{o}_{t-1}, \mathbf{o}_{t+1}, \mathbf{o}_{t+2} \ldots) = P(\mathbf{o}_t | s_t)$$

Hidden Markov Model (cont.)

- Two major types of HMMs according to the observations
  - **Discrete and finite observations:**
    - The observations that all distinct states generate are finite in number
      $\boldsymbol{V}=\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \ldots\ldots, \boldsymbol{v}_M\}, \boldsymbol{v}_k \in \boldsymbol{R}^L$
    - In this case, the set of observation probability distributions $B=\{b_j(\mathbf{v}_\mathrm{k})\}$, is defined as $b_j(\mathbf{v}_\mathrm{k})=P(\mathbf{o}_t=\mathbf{v}_\mathrm{k}|s_t=j)$, $1 \le k \le M$, $1 \le j \le N$
      $\mathbf{o}_t$: *observation at time t*, s_t : *state at time t*
      ⇨ *for state j*, $b_j(\mathbf{v}_\mathrm{k})$ consists of *only M probability values*

A left-to-right HMM



Hidden Markov Model (cont.)

- Two major types of HMMs according to the observations
  - **Continuous and infinite observations:**
    - The observations that all distinct states generate are infinite and continuous, that is, $\boldsymbol{V}=\{\mathbf{v} \mid \mathbf{v}\in \boldsymbol{R}^d\}$
    - In this case, the set of observation probability distributions $B=\{b_j(\mathbf{v})\}$, is defined as $b_j(\mathbf{v})=f_{\mathbf{O}|\mathbf{S}}(\boldsymbol{o}_t=\mathbf{v}|s_t=j)$, $1\leq j\leq N$
      ⇨ $b_j(\mathbf{v})$ is a ***continuous probability density function (pdf) and is often a mixture of Multivariate Gaussian (Normal) Distributions***

$$b_j(\mathbf{v})=\sum_{k=1}^{M} w_{jk}\left(\frac{1}{(2\pi)^{d/2}\left|\boldsymbol{\Sigma}_{jk}\right|^{1/2}}\exp\left(-\frac{1}{2}\left(\mathbf{v}-\boldsymbol{\mu}_{jk}\right)^t\boldsymbol{\Sigma}_{jk}^{-1}\left(\mathbf{v}-\boldsymbol{\mu}_{jk}\right)\right)\right)$$

Mixture Weight

Covariance Matrix

Observation Vector

Mean Vector

Hidden Markov Model (cont.)

- Multivariate Gaussian Distributions
  - When ***X***=(***x***$_1$, ***x***$_2$,…, ***x***$_d$) is a *d*-dimensional random vector, the multivariate Gaussian pdf has the form:

$$f(\mathbf{X}=\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma})=N(\mathbf{x};\boldsymbol{\mu},\boldsymbol{\Sigma})=\frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}}\exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^t\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$

where $\boldsymbol{\mu}$ is the L-dimensional mean vector, $\boldsymbol{\mu}=E[\mathbf{x}]$

$\boldsymbol{\Sigma}$ is the coverance matrix, $\boldsymbol{\Sigma}=E\left[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^t\right]=E\left[\mathbf{x}\mathbf{x}^t\right]-\boldsymbol{\mu}\boldsymbol{\mu}^t$

and $|\boldsymbol{\Sigma}|$ is the the determinant of $\boldsymbol{\Sigma}$

The i-j^{th} elevment σ_{ij} of $\boldsymbol{\Sigma}$, $\sigma_{ij}=E\left[(x_i-\mu_i)(x_j-\mu_j)\right]=E\left[x_i x_j\right]-\mu_i\mu_j$

  - If ***x***$_1$, ***x***$_2$,…, ***x***$_d$ are independent, the covariance matrix is reduced to diagonal covariance
    - Viewed as *d* independent scalar Gaussian distributions
    - Model complexity is significantly reduced

Hidden Markov Model (cont.)

- Multivariate Gaussian Distributions

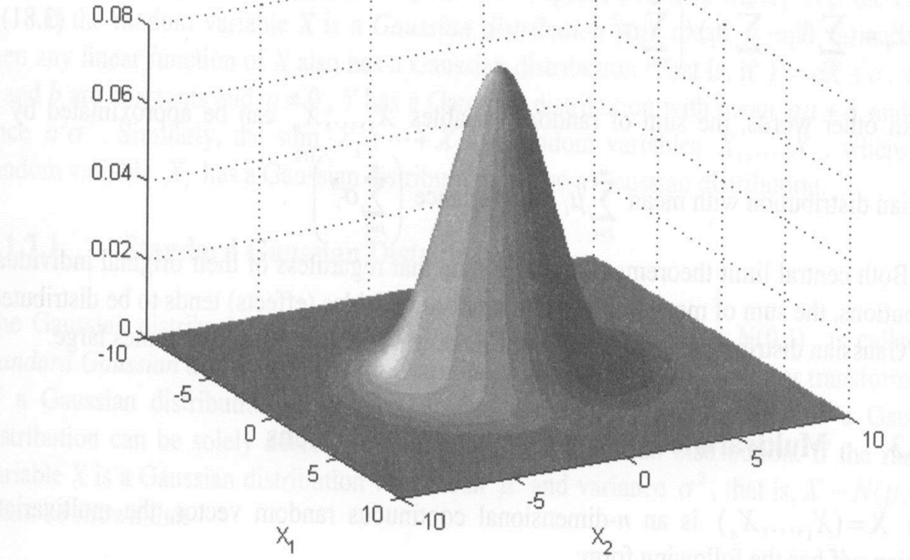


Figure 3.12 A two-dimensional multivariate Gaussian distribution with independent random variables x_1 and x_2 that have the same variance.

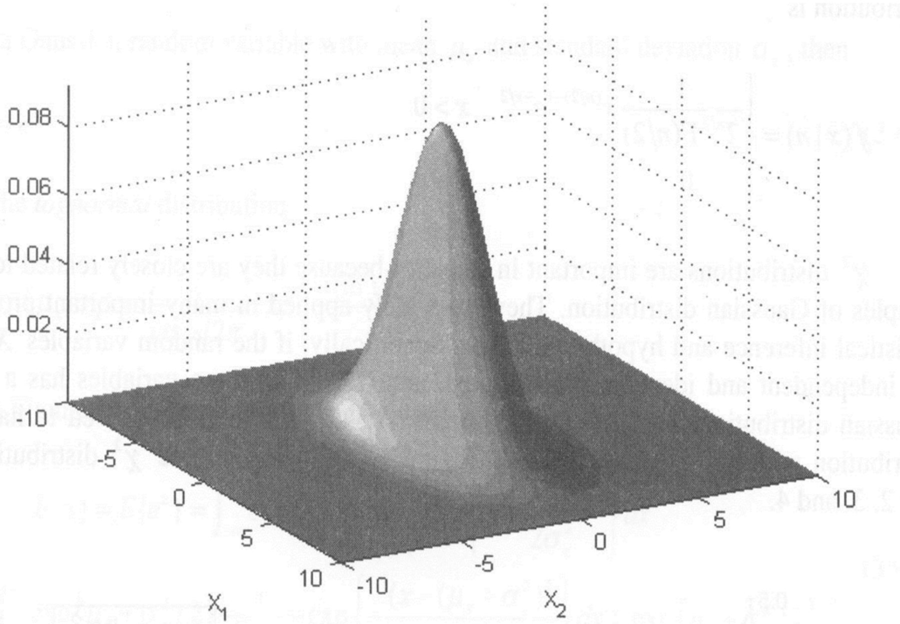
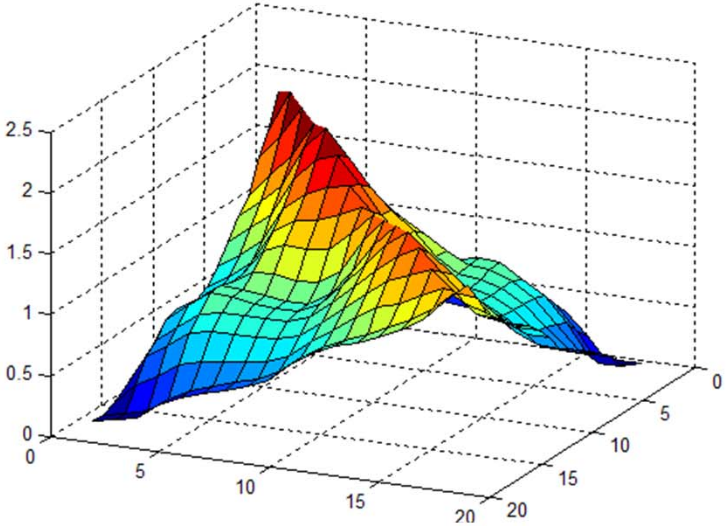


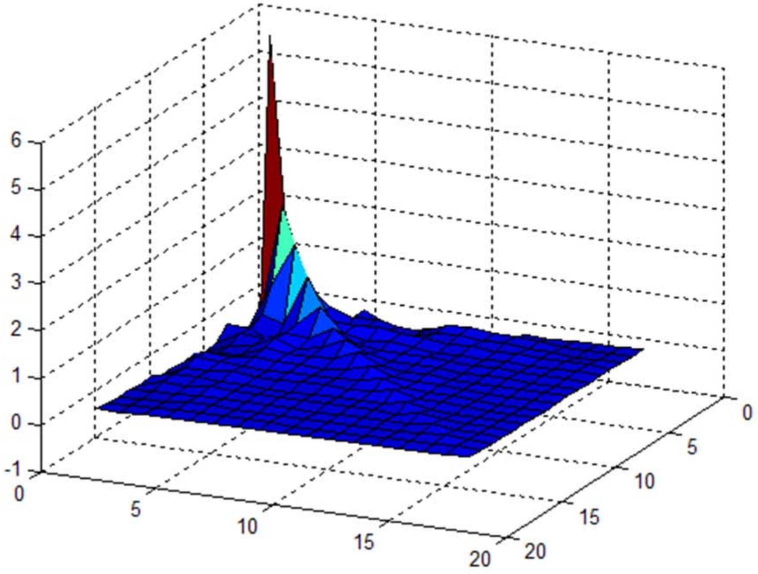
Figure 3.13 Another two-dimensional multivariate Gaussian distribution with independent random variable x_1 and x_2 which have different variances.

Hidden Markov Model (cont.)

- Covariance matrix of the correlated feature vectors (Mel-frequency filter bank outputs)



- Covariance matrix of the partially de-correlated feature vectors (MFCC without C_0)
  - MFCC: Mel-frequency cepstral coefficients

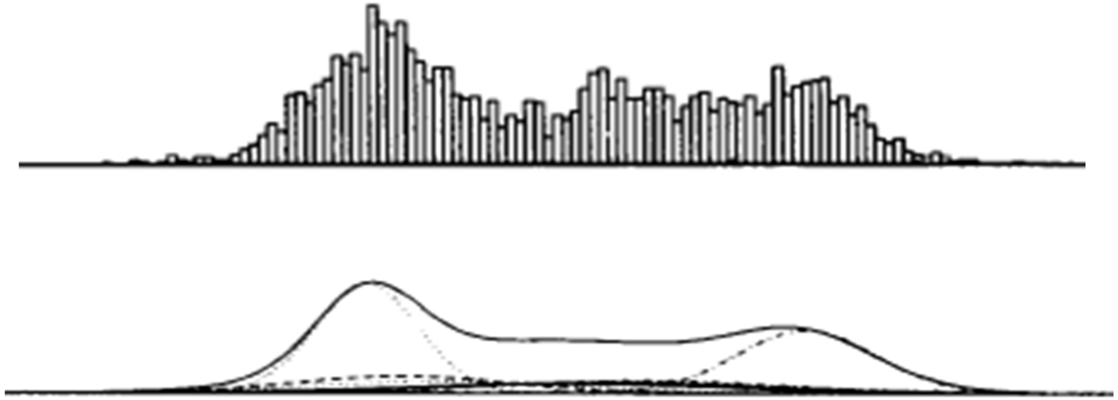


Hidden Markov Model (cont.)

- Multivariate ***Mixture*** Gaussian Distributions (cont.)
  - More complex distributions with multiple local maxima can be approximated by Gaussian (a unimodal distribution) mixture

$$f(\boldsymbol{x}) = \sum_{k=1}^{M} w_k N_k(\boldsymbol{x}; \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k), \qquad \sum_{k=1}^{M} w_k = 1$$

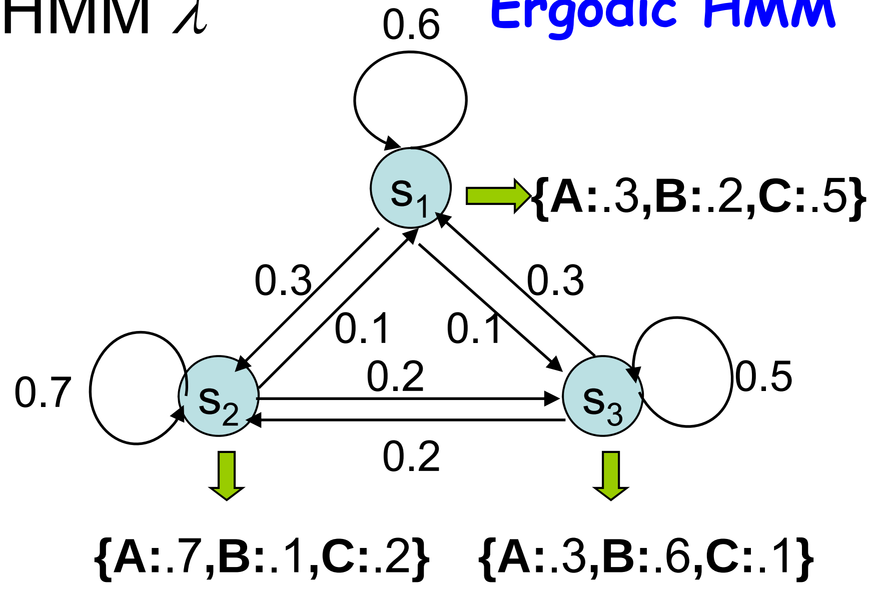
  - Gaussian mixtures with enough mixture components can approximate any distribution



Hidden Markov Model (cont.)

- Example 4: a 3-state discrete HMM λ

Ergodic HMM



$$\mathbf{A} = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

$$b_1(\mathbf{A}) = 0.3, b_1(\mathbf{B}) = 0.2, b_1(\mathbf{C}) = 0.5$$

$$b_2(\mathbf{A}) = 0.7, b_2(\mathbf{B}) = 0.1, b_2(\mathbf{C}) = 0.2$$

$$b_3(\mathbf{A}) = 0.3, b_3(\mathbf{B}) = 0.6, b_3(\mathbf{C}) = 0.1$$

$$\pi = [0.4 \quad 0.5 \quad 0.1]$$

  - Given a sequence of observations *O*={*ABC*}, there are **27 possible** corresponding state sequences, and therefore the corresponding probability is

$$P(\boldsymbol{O}|\lambda) = \sum_{i=1}^{27} P(\boldsymbol{O}, \boldsymbol{S}_i|\lambda) = \sum_{i=1}^{27} P(\boldsymbol{O}|\boldsymbol{S}_i, \lambda) P(\boldsymbol{S}_i|\lambda), \qquad \boldsymbol{S}_i : \text{state sequence}$$

$$E.g. \text{ when } \boldsymbol{S}_i = \{s_2 s_2 s_3\}, P(\boldsymbol{O}|\boldsymbol{S}_i, \boldsymbol{\lambda}) = P(\boldsymbol{A}|s_2) P(\boldsymbol{B}|s_2) P(\boldsymbol{C}|s_3) = 0.7 * 0.1 * 0.1 = 0.007$$

$$P(\boldsymbol{S}_i|\boldsymbol{\lambda}) = P(s_2) P(s_2|s_2) P(s_3|s_2) = 0.5 * 0.7 * 0.2 = 0.07$$

Hidden Markov Model (cont.)

- Notations:
  - ***O*={*o*$_1$*o*$_2$*o*$_3$……*o*$_T$}:** the observation (feature) sequence
  - ***S*= {**$s_1 s_2 s_3 \ldots\ldots s_T$**} :** the state sequence
  - λ : model, for HMM, λ={***A*,*B*,**π}
  - *P*(***O***|λ) : The probability of observing ***O*** given the model λ
  - *P*(***O***|***S*,**λ) : The probability of observing ***O*** given λ and a state sequence ***S*** of λ
  - *P*(***O*,*S***|λ) : The probability of observing ***O*** and ***S*** given λ
  - *P*(***S***|***O*,**λ) : The probability of observing ***S*** given ***O*** and λ
- Useful formulas
  - Bayes' Rule :

$$P(A|B)=\frac{P(A,B)}{P(B)}=\frac{P(B|A)P(A)}{P(B)} \Rightarrow P(A|B,\lambda)=\frac{P(A,B|\lambda)}{P(B|\lambda)}=\frac{P(B|A,\lambda)P(A|\lambda)}{P(B|\lambda)}$$

λ : model describing the probability

$$P(A,B)=P(B|A)P(A)=P(A|B)P(B)$$

chain rule

Hidden Markov Model (cont.)

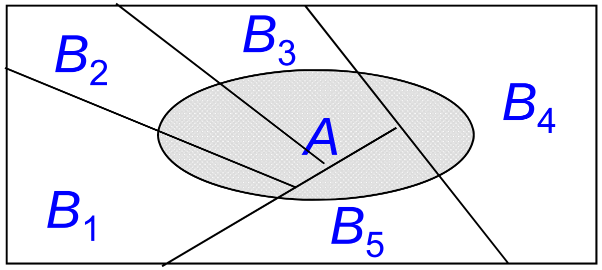
- Useful formulas (Cont.):
  - Total Probability Theorem

marginal probability

$$P(A)=\begin{cases}\sum_{all\ B} P(A,B)=\sum_{all\ B} P(A|B)P(B), & \text{if } B \text{ is disrete and disjoint}\\ \int_B f(A,B)dB=\int_B f(A|B)f(B)dB, & \text{if } B \text{ is continuous}\end{cases}$$

$$\text{if } x_1, x_2, \ldots\ldots, x_n \text{ are independent,}$$

$$\Rightarrow P(x_1, x_2, \ldots\ldots, x_n)=P(x_1)P(x_2)\ldots\ldots P(x_n)$$



Venn Diagram

$$E_z[q(z)]=\begin{cases}\sum_k P(z=k)q(k), & z:\text{discrete}\\ \int_z f_{\mathbf{z}}(z)q(z)dz, & z:\text{continuous}\end{cases}$$

Expectation

Three Basic Problems for HMM

- Given an observation sequence $O=(o_1,o_2,\ldots\ldots,o_T)$, and an HMM $\lambda=(S,A,B,\pi)$
  - **Problem *1***:

    How to *efficiently* compute $P(\mathbf{O}|\lambda)$?

    ⇨ ***Evaluation problem***
  - **Problem *2*:**

    How to choose an optimal state sequence $\mathbf{S}=(s_1,s_2,\ldots\ldots,s_T)$?

    ⇨ ***Decoding Problem***
  - **Problem *3*:**

    How to adjust the model parameter $\boldsymbol{\lambda}=(\mathbf{A},B,\pi)$ to maximize $P(\mathbf{O}|\lambda)$?

    ⇨ ***Learning / Training Problem***

Basic Problem 1 of HMM (cont.)

Given ***O*** and λ, find $P(\boldsymbol{O}|\lambda)$= Prob[observing *O* given λ]

- Direct Evaluation
  - Evaluating all possible state sequences of length *T* that generating observation sequence ***O***

$$P(\boldsymbol{O}|\lambda) = \sum_{all\ \boldsymbol{S}} P(\boldsymbol{O}, \boldsymbol{S}|\lambda) = \sum_{all\ S} P(\boldsymbol{O}|\boldsymbol{S}, \lambda) P(\boldsymbol{S}|\lambda)$$

  - $P(\boldsymbol{S}|\lambda)$: The probability of each path ***S***
    - By Markov assumption (First-order HMM)

$$P(\boldsymbol{S}|\lambda) = P(s_1|\lambda)\prod_{t=2}^{T} P\left(s_t \middle| s_1^{t-1}, \lambda\right)$$

By chain rule

$$\approx P(s_1|\lambda)\prod_{t=2}^{T} P(s_t|s_{t-1}, \lambda)$$

By Markov assumption

$$= \pi_{s_1} a_{s_1 s_2} a_{s_2 s_3} \dots a_{s_{T-1} s_T}$$

Basic Problem 1 of HMM (cont.)

- Direct Evaluation (cont.)
  - $P(\boldsymbol{O}|\boldsymbol{S},\lambda)$: The joint output probability along the path *S*
    - By output-independent assumption
      - The probability that a particular observation symbol/vector is emitted at time *t* depends only on the state s_t and is conditionally independent of the past observations

$$\begin{aligned}P(\boldsymbol{O}|\boldsymbol{S},\lambda) &= P\left(\boldsymbol{o}_1^T \middle| s_1^T,\lambda\right)\\ &= P\left(\boldsymbol{o}_1 \middle| s_1^T,\lambda\right)\prod_{t=2}^{T} P\left(\boldsymbol{o}_t \middle| \boldsymbol{o}_1^{t-1}, s_1^T,\lambda\right)\\ &\approx \prod_{t=1}^{T} P\left(\boldsymbol{o}_t \middle| s_t,\lambda\right)\\ &= \prod_{t=1}^{T} b_{s_t}(\boldsymbol{o}_t)\end{aligned}$$

By output-independent assumption

Basic Problem 1 of HMM (cont.)

- Direct Evaluation (Cont.)

$P(\boldsymbol{o}_t | s_t, \lambda) = b_{s_t}(\boldsymbol{o}_t)$

$$P(\boldsymbol{O}|\lambda) = \sum_{all\ \boldsymbol{S}} P(\boldsymbol{S}|\lambda) P(\boldsymbol{O}|\boldsymbol{S}, \lambda)$$

$$= \sum_{all\ \boldsymbol{s}} \left(\left[\pi_{s_1} a_{s_1 s_2} a_{s_2 s_3} \ldots\ldots a_{s_{T-1} s_T} \right] \left[b_{s_1}(\boldsymbol{o}_1) b_{s_2}(\boldsymbol{o}_2) \ldots\ldots b_{s_T}(\boldsymbol{o}_T) \right] \right)$$

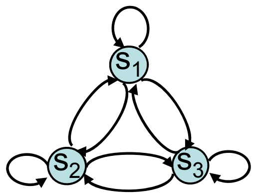
$$= \sum_{s_1, s_2, .., s_T} \pi_{s_1} b_{s_1}(\boldsymbol{o}_1) a_{s_1 s_2} b_{s_2}(\boldsymbol{o}_2) \ldots\ldots a_{s_{T-1} s_T} b_{s_T}(\boldsymbol{o}_T)$$

  - Huge Computation Requirements: $O(N^T)$
    - Exponential computational complexity

$$\textbf{\textit{Complexity}} \quad : (2T\text{-}1) N^T \ MUL \quad \approx 2TN^T, N^T\text{-}1 \text{ ADD}$$

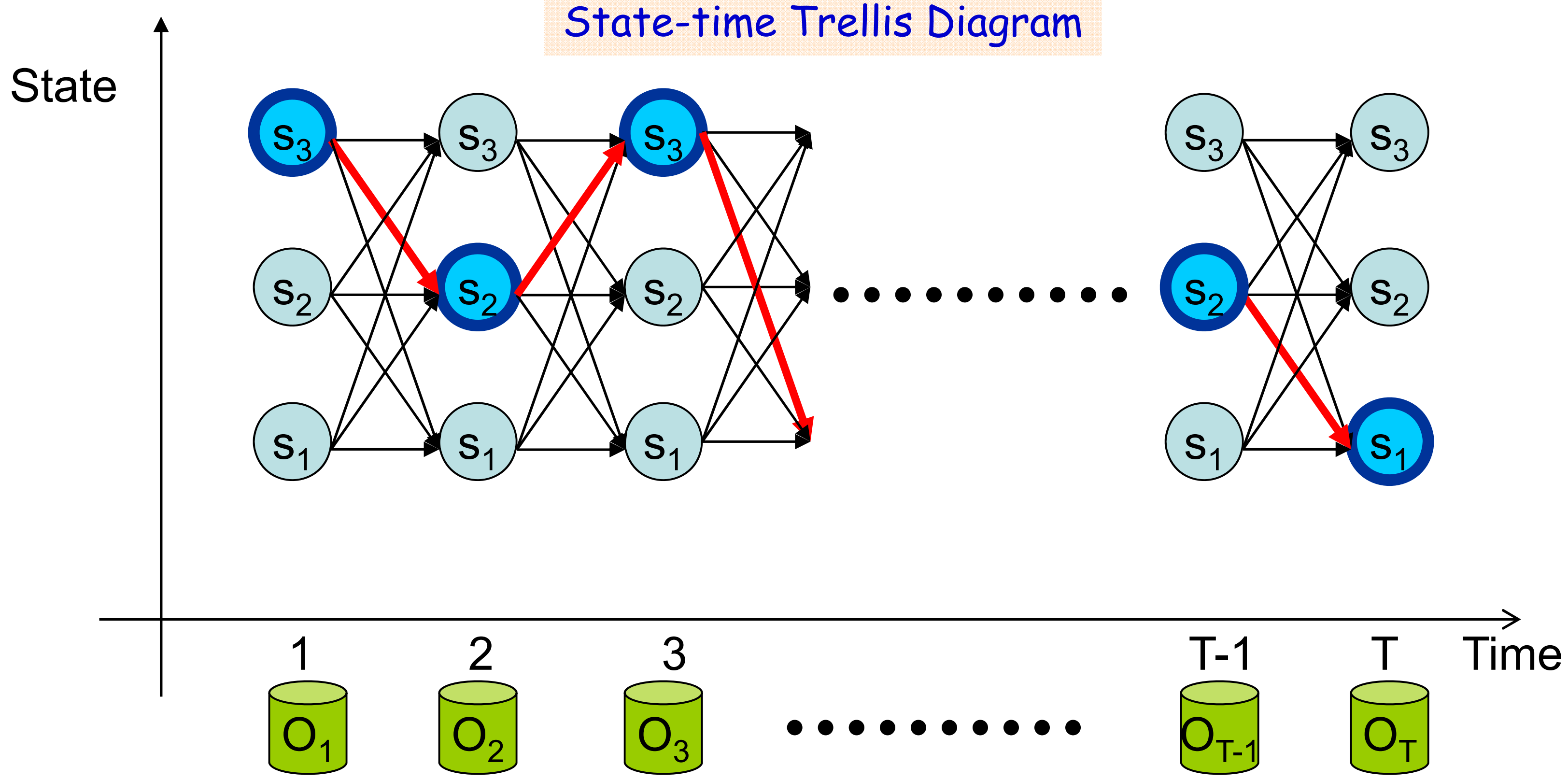
- A more efficient algorithms can be used to evaluate $P(\boldsymbol{O}|\lambda)$
  - *Forward/Backward Procedure/Algorithm*

Basic Problem 1 of HMM (cont.)



- Direct Evaluation (Cont.)

State-time Trellis Diagram



● s_i **means $b_j(o_t)$ has been computed**

a_{ij} → **means a_{ij} has been computed**

Basic Problem 1 of HMM

- The Forward Procedure

- Base on the HMM assumptions, the calculation of $P(s_t|s_{t-1},\lambda)$and $P(\boldsymbol{o}_t|s_t,\lambda)$ involves only s_{t-1}, s_t and $\boldsymbol{o}_t$, so it is possible to compute the likelihood with recursion on t

- Forward variable : $\alpha_t(i) = P(o_1 o_2 \ldots o_t, s_t = i|\boldsymbol{\lambda})$
  - The probability that the HMM is in state *i* at time *t* having generating partial observation $\boldsymbol{o}_1\boldsymbol{o}_2\ldots\boldsymbol{o}_t$

Basic Problem 1 of HMM
- The Forward Procedure (cont.)

- Algorithm

  1. Initialization $\alpha_1(i) = \pi_i b_i(\boldsymbol{o}_1),\ 1 \le i \le N$

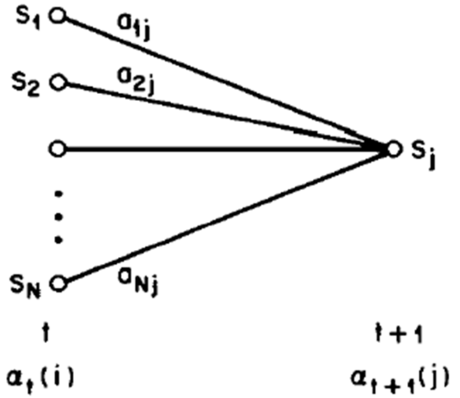
  2. Induction $\alpha_{t+1}(j) = \left[\sum_{i=1}^{N} \alpha_t(i) a_{ij}\right] b_j(\boldsymbol{o}_{t+1}),\ \ 1 \le t \le T\text{-}1, 1 \le j \le N$

  3. Termination $P(\boldsymbol{O}|\lambda) = \sum_{i=1}^{N} \alpha_T(i)$

  – Complexity: $O(N^2T)$

    MUL : $N(N+1)(T-1)+N \quad \approx \quad N^2T$

    ADD : $(N-1)N(T-1)+(N-1) \approx N^2T$



- Based on the lattice (trellis) structure
  – Computed in a *time-synchronous* fashion from *left-to-right*, where each cell for time *t* is completely computed before proceeding to time *t+1*

- All state sequences, regardless how long previously, merge to *N* nodes (states) at each time instance *t*

Basic Problem 1 of HMM

- The Forward Procedure (cont.)

$$\begin{aligned}
\alpha_t(j) &= P(o_1o_2\ldots o_t, s_t = j|\lambda)\\
&= P(o_1o_2\ldots o_t|s_t = j,\lambda)P(s_t = j|\lambda)\\
&= P(o_1o_2\ldots o_{t-1}|s_t = j,\lambda)P(o_t|s_t = j,\lambda)P(s_t = j|\lambda)\\
&= P(o_1o_2\ldots o_{t-1}, s_t = j|\lambda)P(o_t|s_t = j,\lambda)\\
&= P(o_1o_2\ldots o_{t-1}, s_t = j|\lambda)b_j(o_t)\\
&= \left[\sum_{i=1}^{N} P(o_1o_2\ldots o_{t-1}, s_{t-1} = i, s_t = j|\lambda)\right]b_j(o_t)\\
&= \left[\sum_{i=1}^{N} P(o_1o_2\ldots o_{t-1}, s_{t-1} = i|\lambda)P(s_t = j|o_1o_2\ldots o_{t-1}, s_{t-1} = i,\lambda)\right]b_j(o_t)\\
&= \left[\sum_{i=1}^{N} P(o_1o_2\ldots o_{t-1}, s_{t-1} = i|\lambda)P(s_t = j|s_{t-1} = i,\lambda)\right]b_j(o_t)\\
&= \left[\sum_{i=1}^{N} \alpha_{t-1}(i)a_{ij}\right]b_j(o_t)
\end{aligned}$$

$P(A,B)=P(B|A)P(A)$

output independent assumption

$P(B|A)P(A)=P(A,B)$

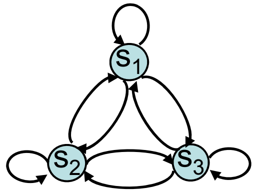
$P(o_t|s_t=j,\lambda)=b_j(o_t)$

$P(A)\sum_{all\,B}P(A,B)$

first-order Markov assumption

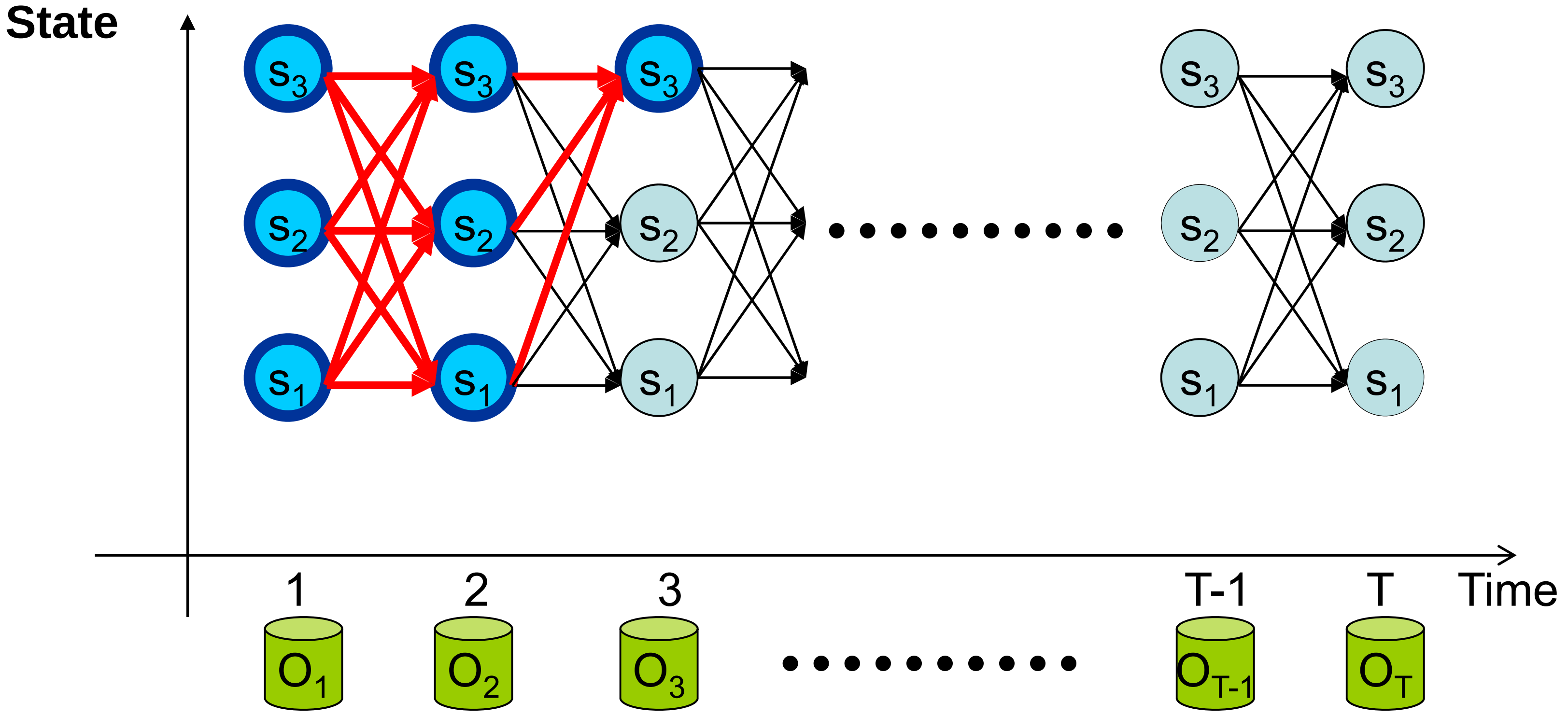
Basic Problem 1 of HMM

- The Forward Procedure (cont.)



- $\alpha_3(3)=P(\boldsymbol{o}_1,\boldsymbol{o}_2,\boldsymbol{o}_3,s_3=3|\lambda)$

  $=[\alpha_2(1)*a_{13}+\alpha_2(2)*a_{23}+\alpha_2(3)*a_{33}]b_3(\boldsymbol{o}_3)$

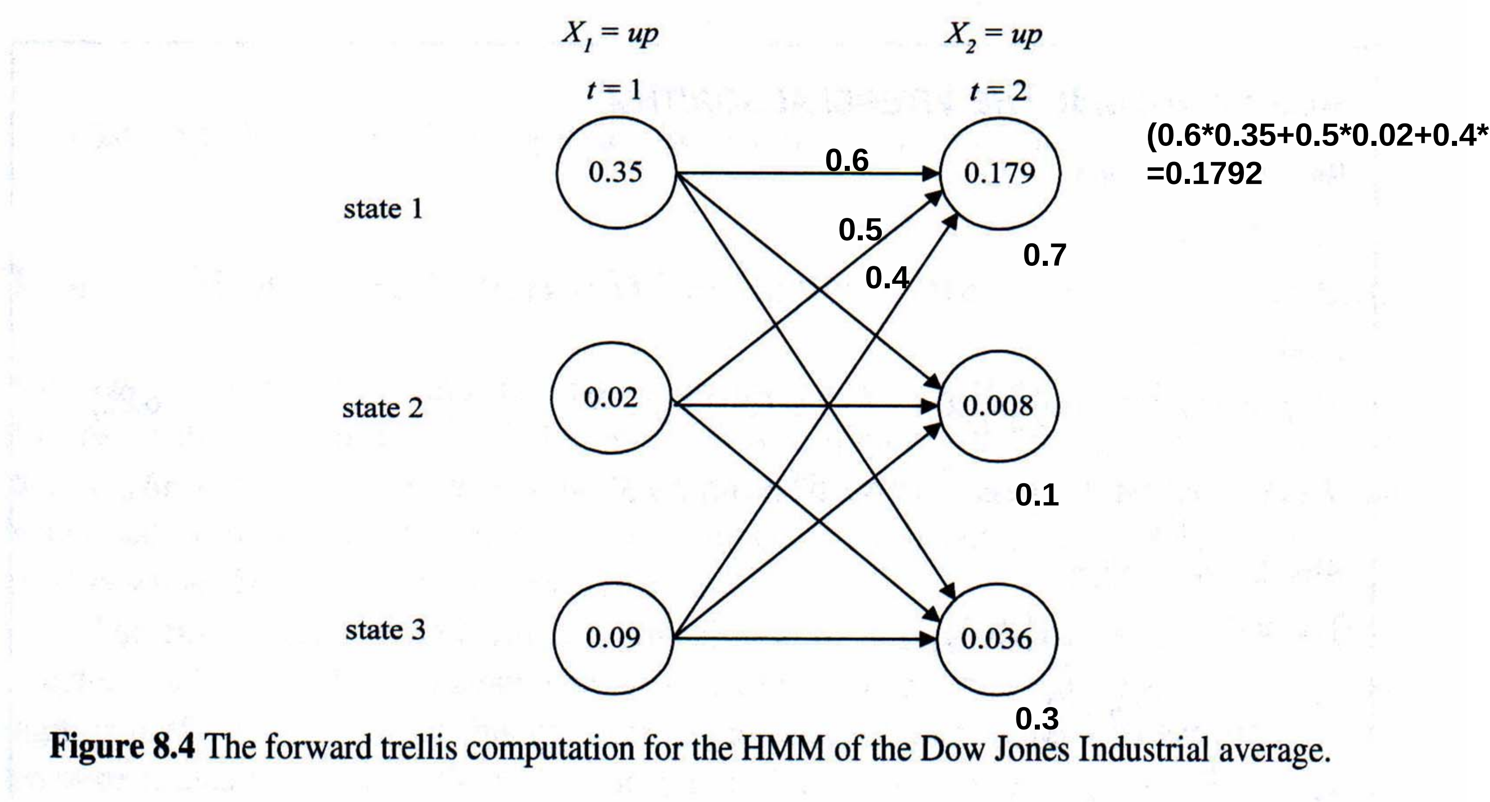


s_i **means $b_j(o_t)$ has been computed**

a_{ij} **means a_{ij} has been computed**

Basic Problem 1 of HMM
- The Forward Procedure (cont.)

- A three-state Hidden Markov Model for the *Dow Jones Industrial average*



(0.6*0.35+0.5*0.02+0.4*0.009)*0.7
=0.1792

Figure 8.4 The forward trellis computation for the HMM of the Dow Jones Industrial average.

Basic Problem 1 of HMM
- The Backward Procedure

- Backward variable : $\beta_t(i)=P(\boldsymbol{o}_{t+1},\boldsymbol{o}_{t+2},\ldots..,\boldsymbol{o}_T|s_t=i\ ,\ \lambda)$

$$1.\ \text{Initialization}: \beta_{\mathrm{T}}(i)=1,\ 1\leq i\leq N$$

$$2.\ \text{Induction}: \beta_t(i)=\sum_{j=1}^{N} a_{ij}b_j(\boldsymbol{o}_{t+1})\beta_{t+1}(j),\ 1\leq t\leq T\text{-}1, 1\leq i\leq N$$

$$3.\ \text{Termination}: P(\boldsymbol{O}|\lambda)=\sum_{j=1}^{N}\pi_j b_j(\boldsymbol{o}_1)\beta_1(j)$$

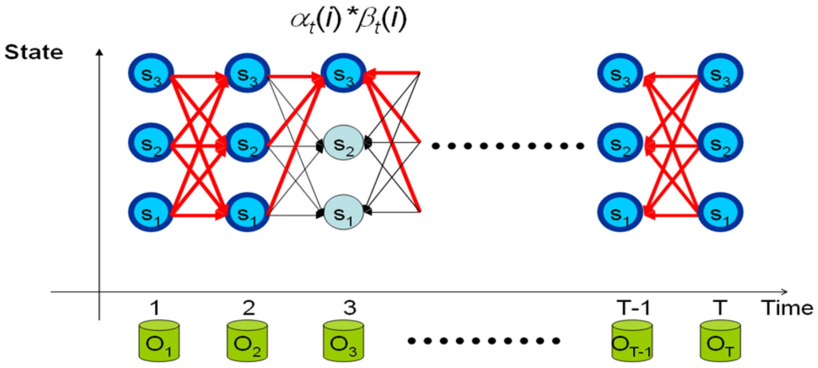
$$\text{Complexity MUL}: 2N^2(T\text{-}1)+2N\approx N^2T\ ;$$

$$\text{ADD}: (N\text{-}1)N(T\text{-}1)+N\approx N^2T$$

Basic Problem 1 of HMM
- Backward Procedure (cont.)

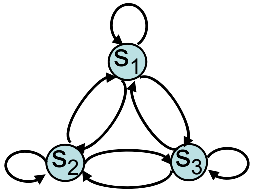
- Why $P(\boldsymbol{O}, s_t = i|\lambda) = \alpha_t(i)\beta_t(i)$?

$$\begin{aligned}
&\alpha_t(i)\beta_t(i)\\
&= P(\boldsymbol{o}_1, \boldsymbol{o}_2, \ldots, \boldsymbol{o}_t, s_t = i|\lambda) \cdot P(\boldsymbol{o}_{t+1}, \boldsymbol{o}_{t+2}, \ldots, \boldsymbol{o}_T|s_t = i, \lambda)\\
&= P(\boldsymbol{o}_1, \boldsymbol{o}_2, \ldots, \boldsymbol{o}_t|s_t = i, \lambda)P(s_t = i|\lambda)P(\boldsymbol{o}_{t+1}, \boldsymbol{o}_{t+2}, \ldots, \boldsymbol{o}_T|s_t = i, \lambda)\\
&= P(\boldsymbol{o}_1, \ldots, \boldsymbol{o}_t, \ldots, \boldsymbol{o}_T|s_t = i, \lambda)P(s_t = i|\lambda)\\
&= P(\boldsymbol{o}_1, \ldots, \boldsymbol{o}_t, \ldots, \boldsymbol{o}_T, s_t = i|\lambda)\\
&= P(\boldsymbol{O}, s_t = i|\lambda)
\end{aligned}$$

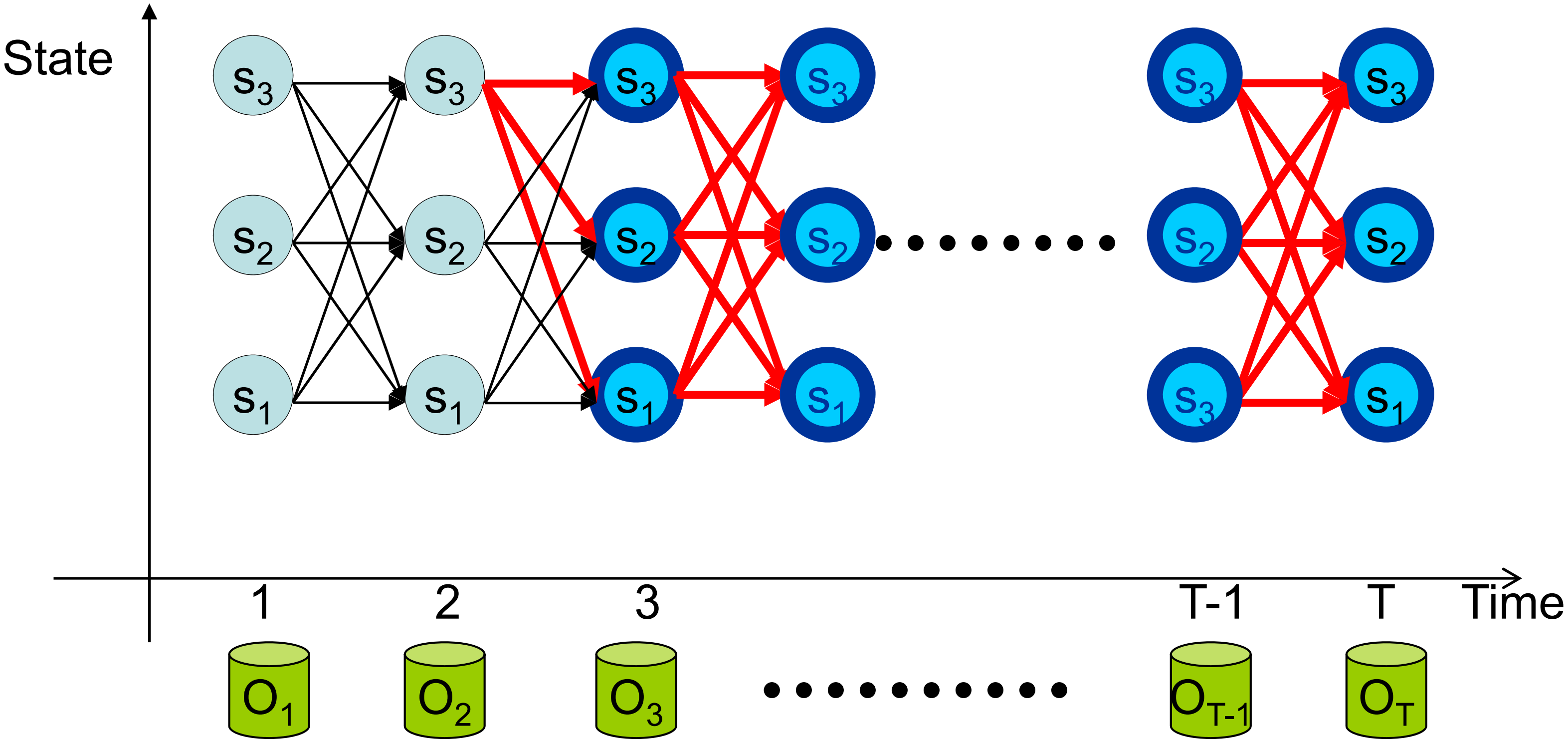


- $P(\boldsymbol{O}|\lambda) = \sum_{i=1}^{N} P(\boldsymbol{O}, s_t = i|\lambda) = \sum_{i=1}^{N} \alpha_t(i)\beta_t(i)$

Basic Problem 1 of HMM
- The Backward Procedure (cont.)



- $\beta_2(3)=P(o_3,o_4,\ldots,\ o_T|s_2=3,\lambda)$
  $=a_{31}*\ b_1(o_3)*\beta_3(1)\ +a_{32}*\ b_2(o_3)*\beta_3(2)+a_{33}*\ b_1(o_3)*\beta_3(3)$



Basic Problem 2 of HMM

How to choose an optimal state sequence $S=(s_1,s_2,\ldots\ldots, s_T)$?

- The first optimal criterion: **Choose the states s_t are *individually* most likely at each time *t***

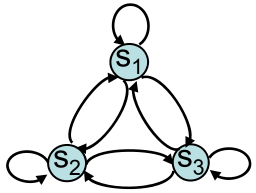
  Define a posteriori probability variable $\gamma_t(i)=P(s_t=i|\boldsymbol{O},\boldsymbol{\lambda})$

$$\gamma_t(i)=\frac{P(s_t=i,\boldsymbol{O}|\boldsymbol{\lambda})}{P(\boldsymbol{O}|\boldsymbol{\lambda})}=\frac{P(s_t=i,\boldsymbol{O}|\boldsymbol{\lambda})}{\sum_{m=1}^{N}P(s_t=m,\boldsymbol{O}|\boldsymbol{\lambda})}=\frac{\alpha_t(i)\,\beta_t(i)}{\sum_{m=1}^{N}\alpha_t(m)\,\beta_t(m)}$$

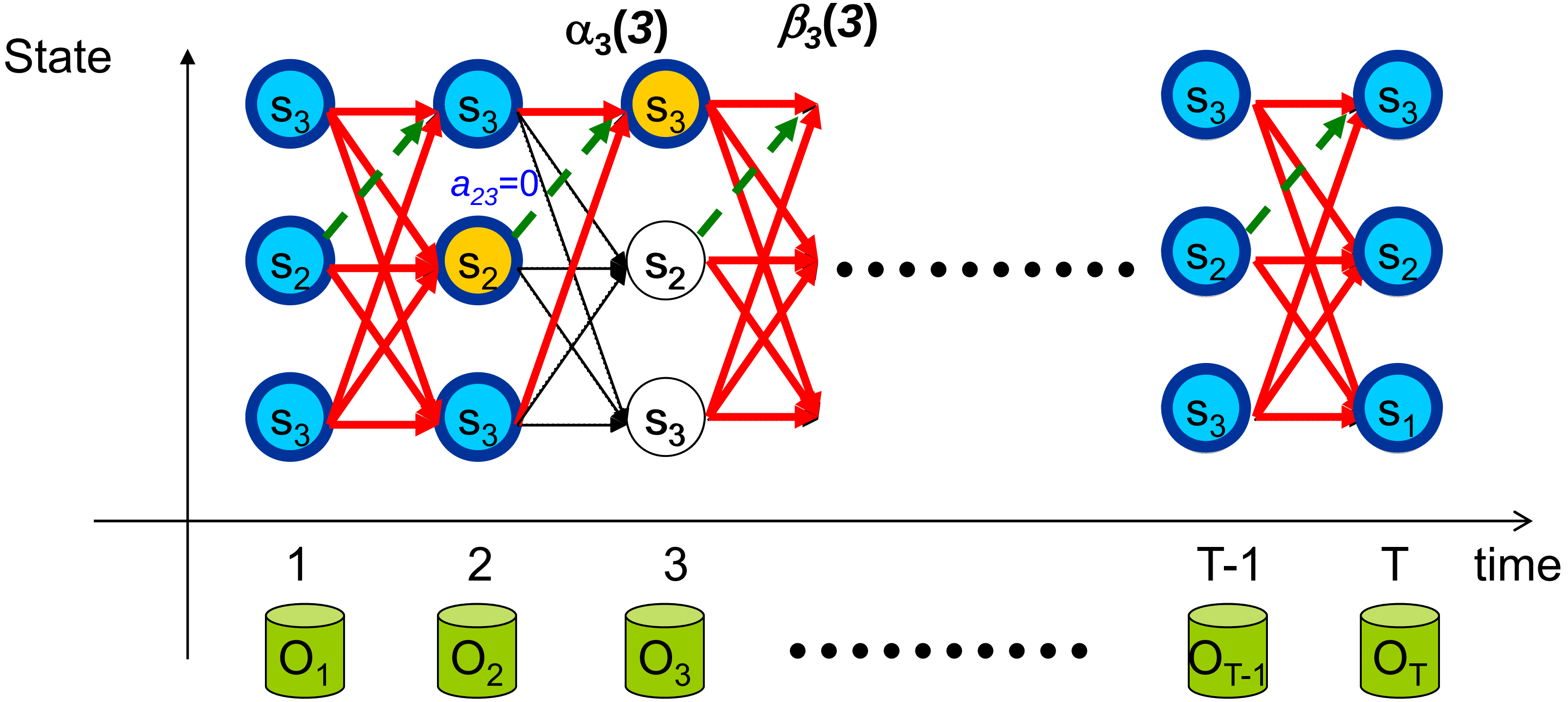
state occupation probability (count) – a soft alignment of HMM state to the observation (feature)

  - Solution : $s_t^* = arg_i\ max\ [\gamma_t(i)],\ 1 \leq t \leq T$
    - Problem: maximizing the probability at each time *t* individually $\boldsymbol{S}^*= s_1^* s_2^* \ldots s_T^*$ may not be a valid sequence (e.g. $a_{s_t^* s_{t+1}^*} = 0$)

Basic Problem 2 of HMM (cont.)



- $P(s_3 = 3, \mathbf{O} \mid \lambda) = \alpha_3(3) * \beta_3(3)$



Basic Problem 2 of HMM
- The Viterbi Algorithm

- The second optimal criterion: The Viterbi algorithm can be regarded as the dynamic programming algorithm applied to the HMM or as a modified forward algorithm

  - Instead of summing up probabilities from different paths coming to the same destination state, the Viterbi algorithm picks and remembers the best path
    - Find a single optimal state sequence $\mathbf{S}=(s_1,s_2,\ldots\ldots, s_T)$
      - How to find the second, third, etc., optimal state sequences (difficult ?)

  - The Viterbi algorithm also can be illustrated in a trellis framework similar to the one for the forward algorithm
    - State-time trellis diagram

Basic Problem 2 of HMM
- The Viterbi Algorithm (cont.)

- Algorithm

Find a best state sequence $\boldsymbol{S}=(s_1,s_2,..,s_T)$ for a given observation $\boldsymbol{O}=(\boldsymbol{o}_1,\boldsymbol{o}_2,..,\boldsymbol{o}_T)$?

Define a new variable

$$\delta_t(i)=\max_{s_1,s_2,..,s_{t-1}} P\left[s_1,s_2,..,s_{t-1},s_t=i,\boldsymbol{o}_1,\boldsymbol{o}_2,..,\boldsymbol{o}_t \middle| \lambda\right]$$

= the best score along a single path at time t, which accounts for the first t observation and ends in state i

By induction $\therefore \delta_{t+1}(j)=\left[\max_{1\le i\le N}\delta_t(i)a_{ij}\right]b_j(\boldsymbol{o}_{t+1})$

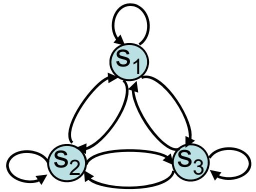
$$\psi_{t+1}(j)=\arg\max_{1\le i\le N}\delta_t(i)a_{ij} \quad \text{.... For backtracing}$$

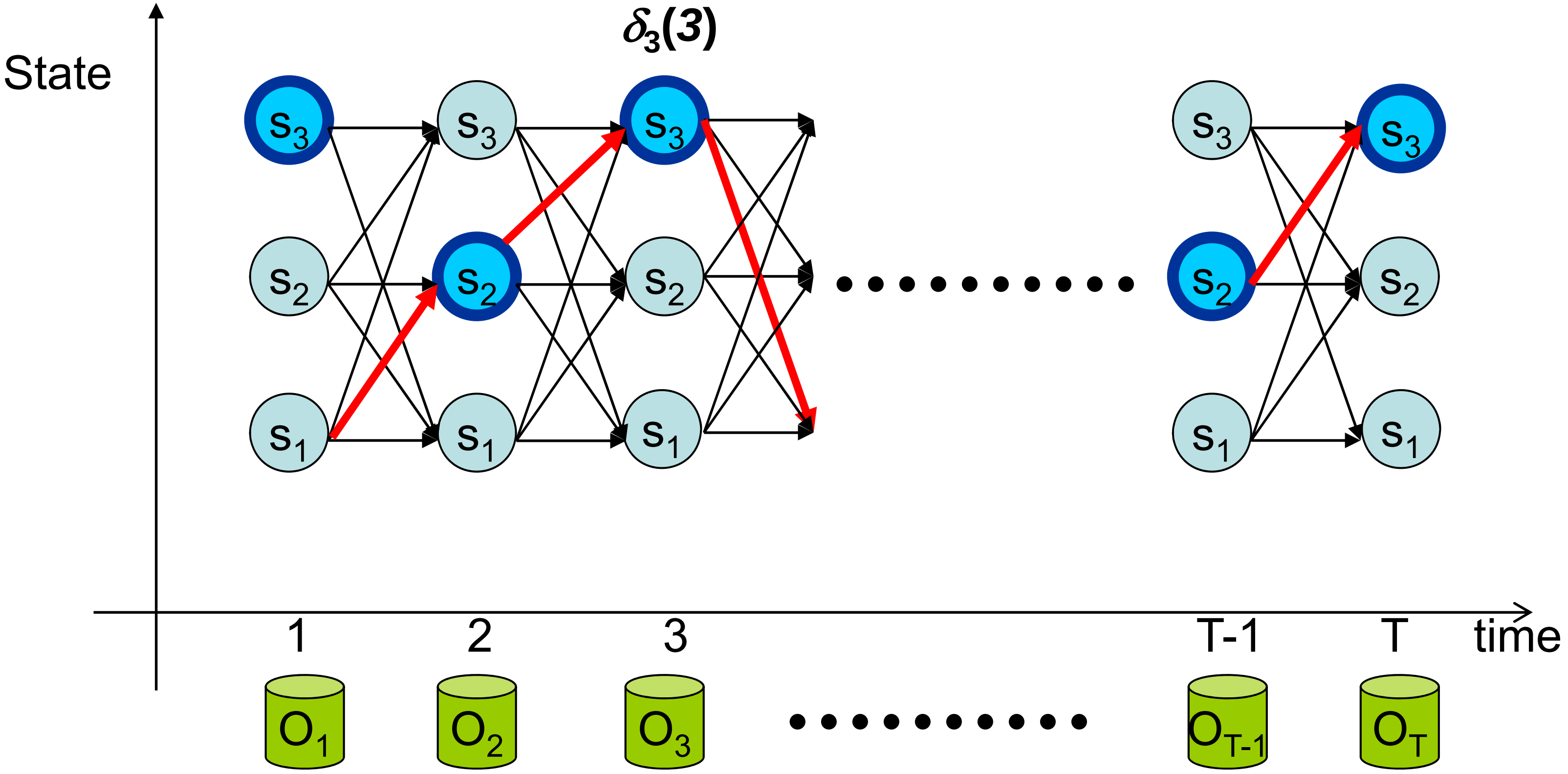
We can backtrace from $s_T^*=\arg\max_{1\le i\le N}\delta_T(i)$

  - Complexity: $O(N^2T)$

Basic Problem 2 of HMM

- The Viterbi Algorithm (cont.)





Basic Problem 2 of HMM
- The Viterbi Algorithm (cont.)

- A three-state Hidden Markov Model for the *Dow Jones Industrial average*

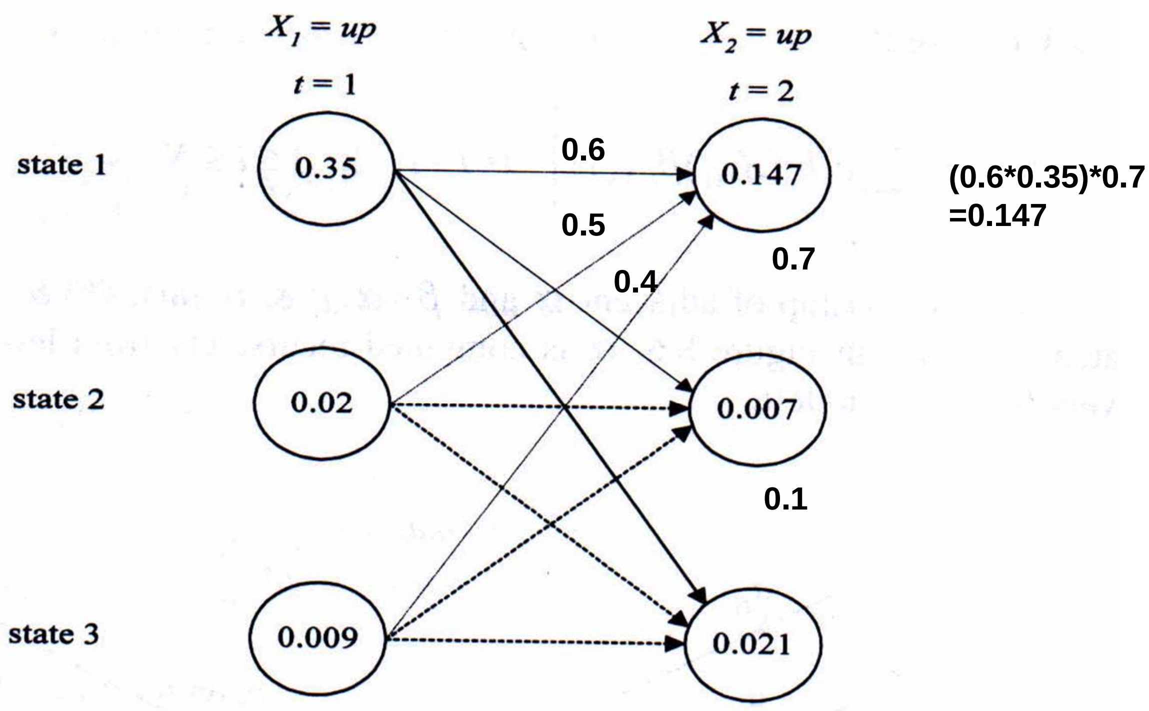


Figure 8.5 The Viterbi trellis computation for the HMM of the Dow Jones Industrial average.

Basic Problem 2 of HMM

- The Viterbi Algorithm (cont.)

- Algorithm in the logarithmic form

Find a best state sequence $\boldsymbol{S}=\left(s_1,s_2,..,s_T\right)$ for a given observation $\boldsymbol{O}=\left(\boldsymbol{o}_1,\boldsymbol{o}_2,..,\boldsymbol{o}_T\right)$?

Define a new variable

$$\delta_t(i)=\max_{s_1,s_2,..,s_{t-1}} \log P\left[s_1,s_2,..,s_{t-1},s_t=i,\boldsymbol{o}_1,\boldsymbol{o}_2,..,\boldsymbol{o}_t \middle| \lambda\right]$$

= the best score along a single path at time t, which accounts for the first t observation and ends in state i

By induction $\therefore \delta_{t+1}(j)=\left[\max_{1\le i\le N}\left(\delta_t(i)+\log a_{ij}\right)\right]+\log b_j\left(\boldsymbol{o}_{t+1}\right)$

$$\psi_{t+1}(j)=\arg\max_{1\le i\le N}\left(\delta_t(i)+\log a_{ij}\right)\ \text{.... For backtracing}$$

We can backtrace from $s_T^*=\arg\max_{1\le i\le N}\delta_T(i)$

Exercise

- A three-state Hidden Markov Model for the *Dow Jones Industrial average*

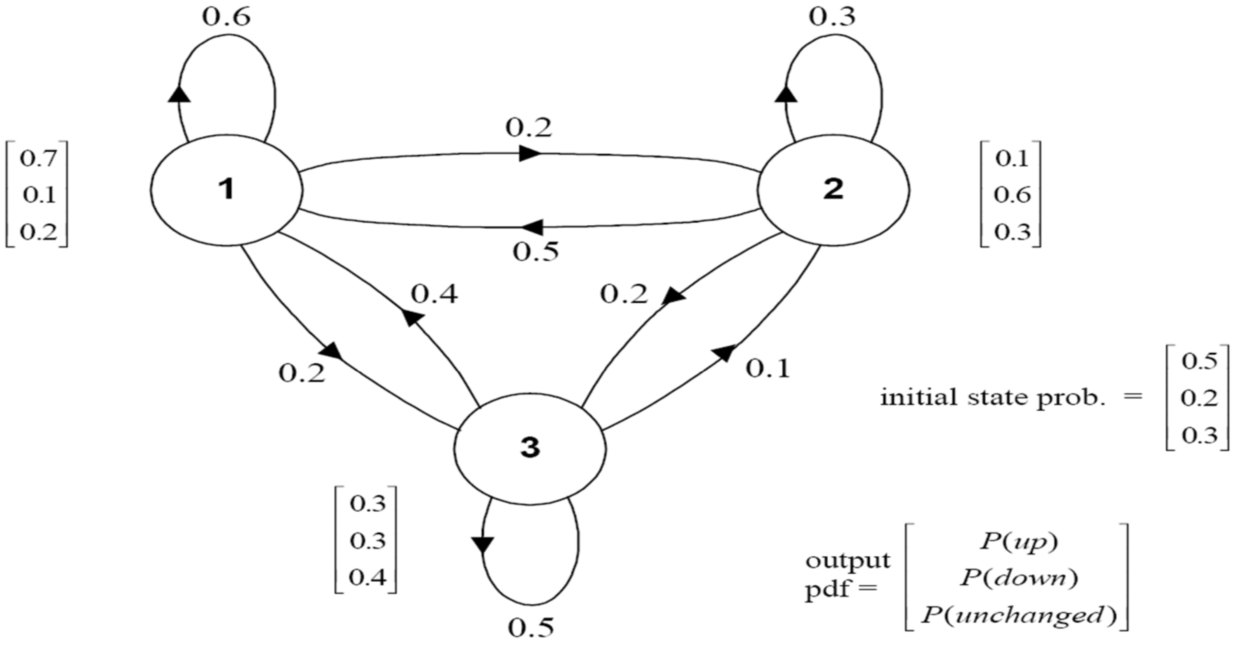
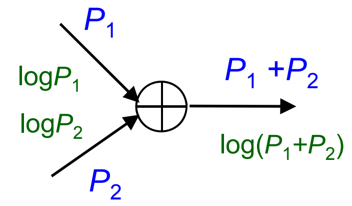


Figure 8.2 A hidden Markov model for the Dow Jones Industrial average. The three states no longer have deterministic meanings as the Markov chain illustrated in Figure 8.1.

  - *Find the probability:*
    P(*up, up, unchanged, down, unchanged, down, up*|λ)
  - *Fnd the optimal state sequence of the model which generates the observation sequence:* (*up, up, unchanged, down, unchanged, down, up*)

Probability Addition in F-B Algorithm

- In Forward-backward algorithm, operations usually implemented in logarithmic domain



- Assume that we want to add P_1 and P_2

$$\text{if } P_1 \geq P_2$$

$$\log_b(P_1 + P_2) = \log P_1 + \log_b\left(1 + b^{\log_b P_2 - \log_b P_1}\right)$$

$$\text{else}$$

$$\log_b(P_1 + P_2) = \log P_2 + \log_b\left(1 + b^{\log_b P_1 - \log_b P_2}\right)$$

The values of $\log_b(1 + b^x)$ can be saved in in a table to speedup the operations

Probability Addition in F-B Algorithm (cont.)

- An example code

```
#define LZERO  (-1.0E10)   // ~log(0)
#define LSMALL (-0.5E10)   // log values < LSMALL are set to LZERO
#define minLogExp  -log(-LZERO) // ~=-23
double LogAdd(double x, double y)
{
 double temp,diff,z;
   if (x<y)
   {
      temp = x; x = y; y = temp;
   }
   diff = y-x; //notice that diff <= 0
   if (diff<minLogExp) // if y' is far smaller than x'
      return  (x<LSMALL) ?  LZERO:x;
   else
    {
      z = exp(diff);
      return x+log(1.0+z);
   }
}
```

Basic Problem 3 of HMM

Intuitive View

- How to adjust (re-estimate) the model parameter λ=(*A*,*B*,π) to maximize $P(\mathbf{O}_1,\ldots,\mathbf{O}_L|\lambda)$ or $\log P(\mathbf{O}_1,\ldots,\mathbf{O}_L|\lambda)$?
  - Belonging to a typical problem of "inferential statistics"
  - The most difficult of the three problems, because there is no known analytical method that maximizes the joint probability of the training data in a close form

$$\log P(\mathbf{O}_1,\mathbf{O}_2,\ldots,\mathbf{O}_L|\lambda)=\log\prod_{l=1}^{L}P(\mathbf{O}_l|\lambda)$$

$$=\sum_{l=1}^{L}\log P(\mathbf{O}_l|\lambda)=\sum_{l=1}^{R}\boxed{\log\sum_{all\,\mathbf{S}}P(\mathbf{S}|\lambda)P(\mathbf{O}_l|\mathbf{S},\lambda)}$$

The "log of sum" form is difficult to deal with

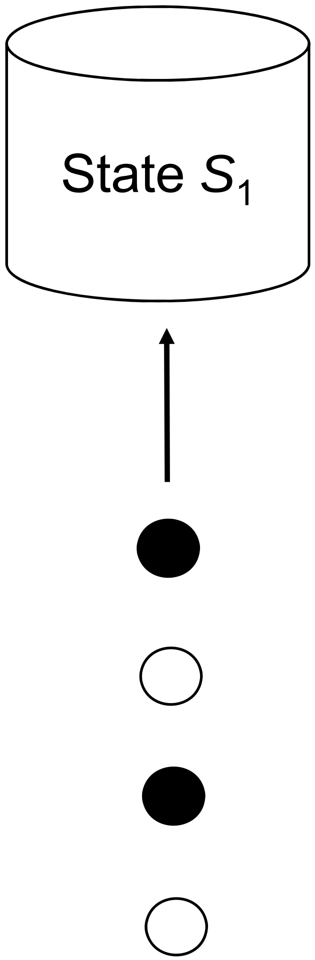
-Suppose that we have L training utterances for the HMM

-$\mathbf{S}$: a possible state sequence of the HMM

  - The data is incomplete because of the hidden state sequences
  - Well-solved by the *Baum-Welch* (known as *forward-backward*) algorithm and *EM* (*Expectation-Maximization*) algorithm
    - Iterative update and improvement
    - Based on Maximum Likelihood (ML) criterion

Maximum Likelihood (ML) Estimation: A Schematic Depiction (1/2)

- Hard Assignment
  - Given the data follow a multinomial distribution

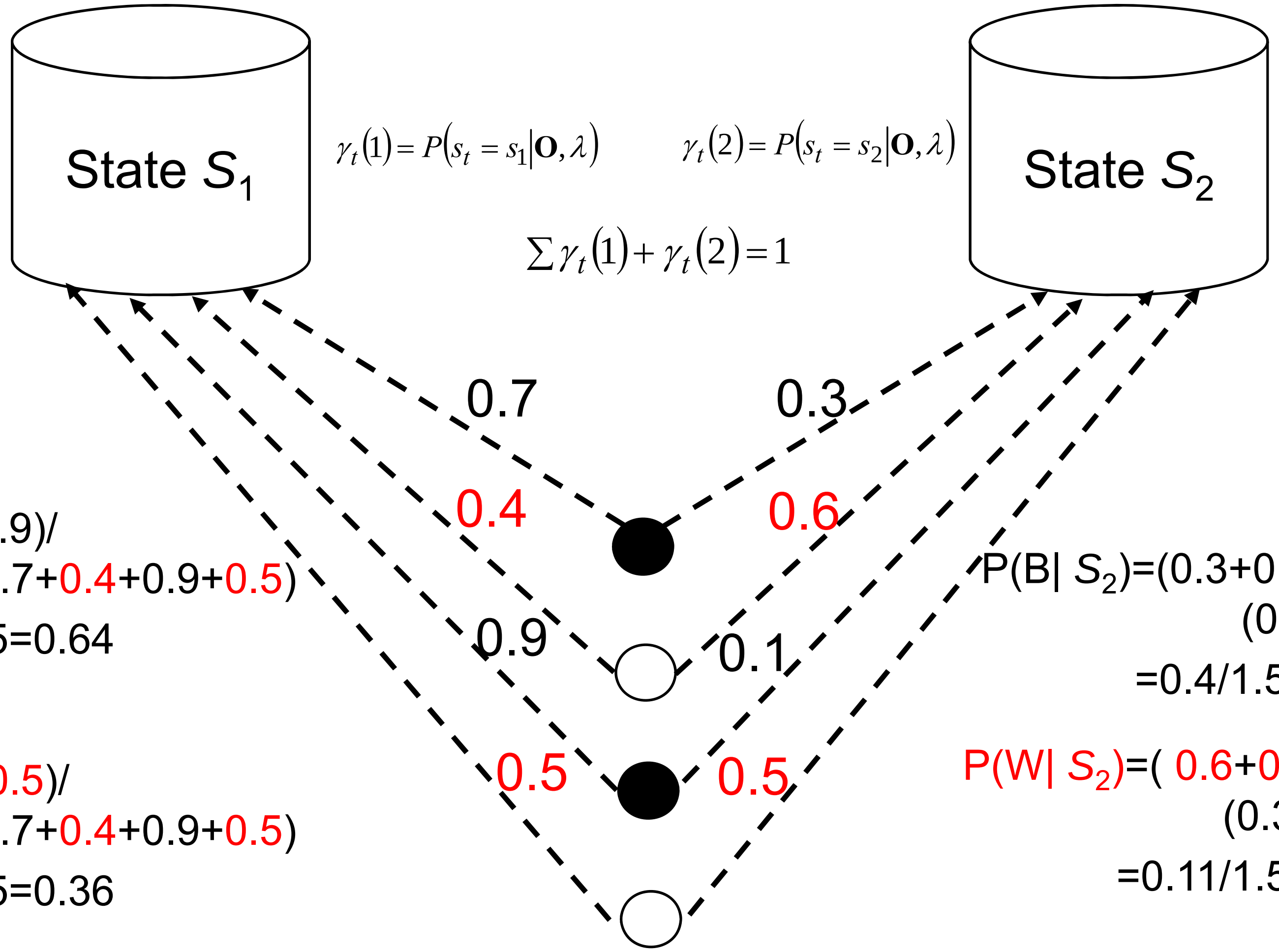


$P(B|\ S_1)=2/4=0.5$

$P(W|\ S_1)=2/4=0.5$

Maximum Likelihood (ML) Estimation: A Schematic Depiction (1/2)

- Soft Assignment
  - Given the data follow a multinomial distribution
  - Maximize the likelihood of the data given the alignment



$\gamma_t(1) = P(s_t = s_1 | \mathbf{O}, \lambda)$ $\gamma_t(2) = P(s_t = s_2 | \mathbf{O}, \lambda)$

$\sum \gamma_t(1) + \gamma_t(2) = 1$

P(B| S_1)=(0.7+0.9)/
(0.7+0.4+0.9+0.5)
=1.6/2.5=0.64

P(W| S_1)=(0.4+0.5)/
(0.7+0.4+0.9+0.5)
=0.9/2.5=0.36

P(B| S_2)=(0.3+0.1)/
(0.3+0.6+0.1+0.5)
=0.4/1.5=0.27

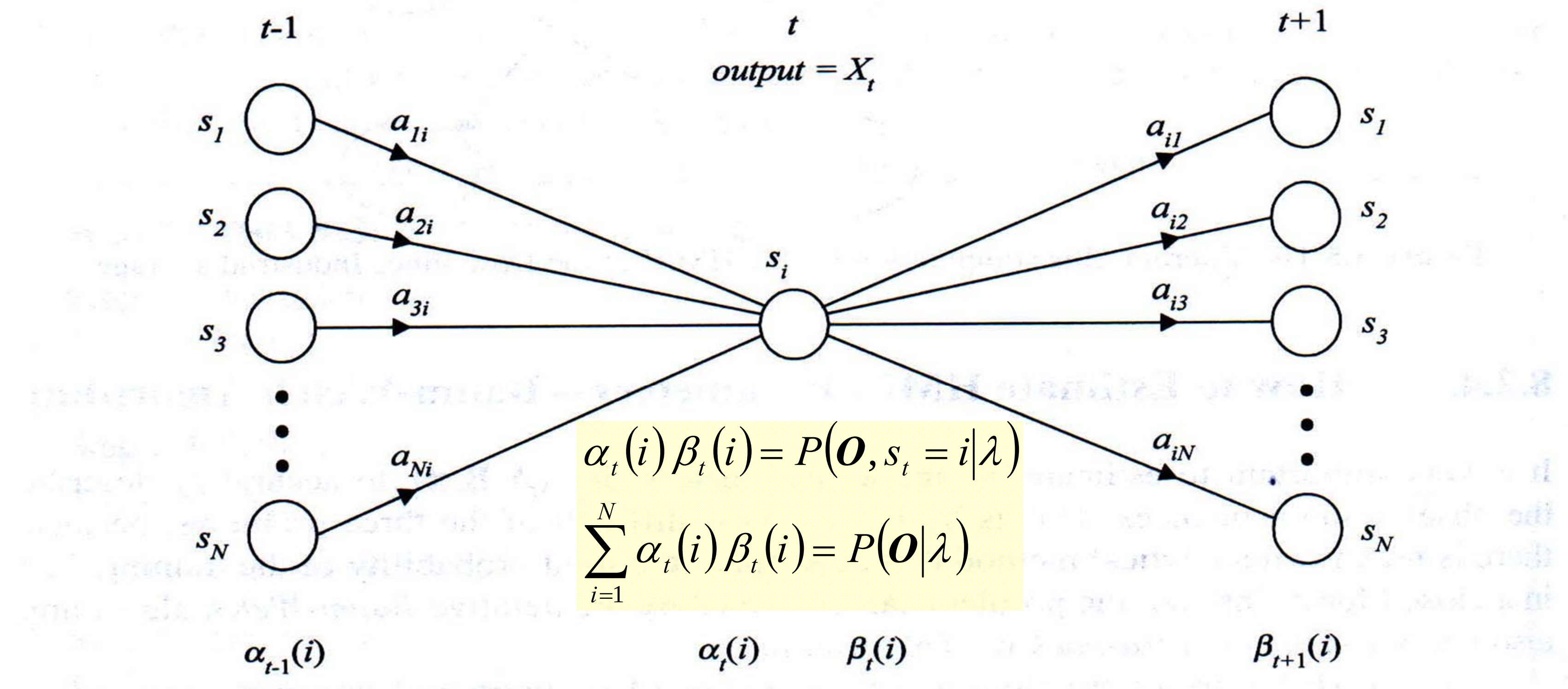
P(W| S_2)=(0.6+0.5)/
(0.3+0.6+0.1+0.5)
=0.11/1.5=0.73

Basic Problem 3 of HMM
Intuitive View (cont.)

- Relationship between the forward and backward variables

$$\alpha_t(i) = P(\boldsymbol{o}_1, \boldsymbol{o}_2, \dots, \boldsymbol{o}_t, s_t = i \mid \lambda)$$
$$= \left[\sum_{j=1}^{N} \alpha_{t-1}(j) a_{ji}\right] b_i(\boldsymbol{o}_t)$$

$$\beta_t(i) = P(\boldsymbol{o}_{t+1}, \boldsymbol{o}_{t+2}, \dots, \boldsymbol{o}_T \mid s_t = i, \lambda)$$
$$= \sum_{j=1}^{N} \beta_{t+1}(j) b_j(\boldsymbol{o}_{t+1}) a_{ij}$$

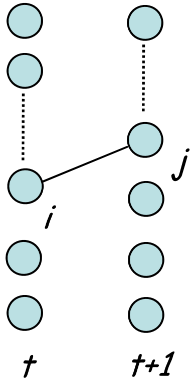


$$\alpha_t(i)\,\beta_t(i) = P(\boldsymbol{O}, s_t = i \mid \lambda)$$
$$\sum_{i=1}^{N} \alpha_t(i)\,\beta_t(i) = P(\boldsymbol{O} \mid \lambda)$$

Figure 8.6 The relationship of α_{t-1} and α_t and β_t and β_{t+1} in the forward-backward algorithm.

Basic Problem 3 of HMM

Intuitive View (cont.)



- Define a new variable:

$$\xi_t(i,j) = P(s_t = i, s_{t+1} = j | \boldsymbol{O}, \boldsymbol{\lambda})$$

  - Probability being at state *i* at time *t* and at state *j* at time *t+1*

$$p(A|B) = \frac{p(A,B)}{P(B)}$$

$$\xi_t(i,j) = \frac{P(s_t = i, s_{t+1} = j, \boldsymbol{O} | \boldsymbol{\lambda})}{P(\boldsymbol{O}|\boldsymbol{\lambda})}$$

$$= \frac{\alpha_t(i) a_{ij} b_j(\boldsymbol{o}_{t+1}) \beta_{t+1}(j)}{P(\boldsymbol{O}|\boldsymbol{\lambda})} = \frac{\alpha_t(i) a_{ij} b_j(\boldsymbol{o}_{t+1}) \beta_{t+1}(j)}{\sum_{m=1}^{N} \sum_{n=1}^{N} \alpha_t(m) a_{mn} b_n(\boldsymbol{o}_{t+1}) \beta_{t+1}(n)}$$

- Recall the posteriori probability variable:

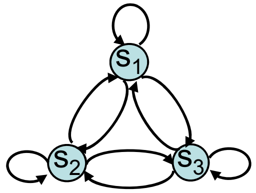
$$\gamma_t(i) = P(s_t = i | \boldsymbol{O}, \lambda)$$

Note : $\gamma_t(i)$ also can be represente d as $\frac{\alpha_t(i)\beta_t(i)}{\sum_{m=1}^{N} \alpha_t(m)\beta_t(m)}$

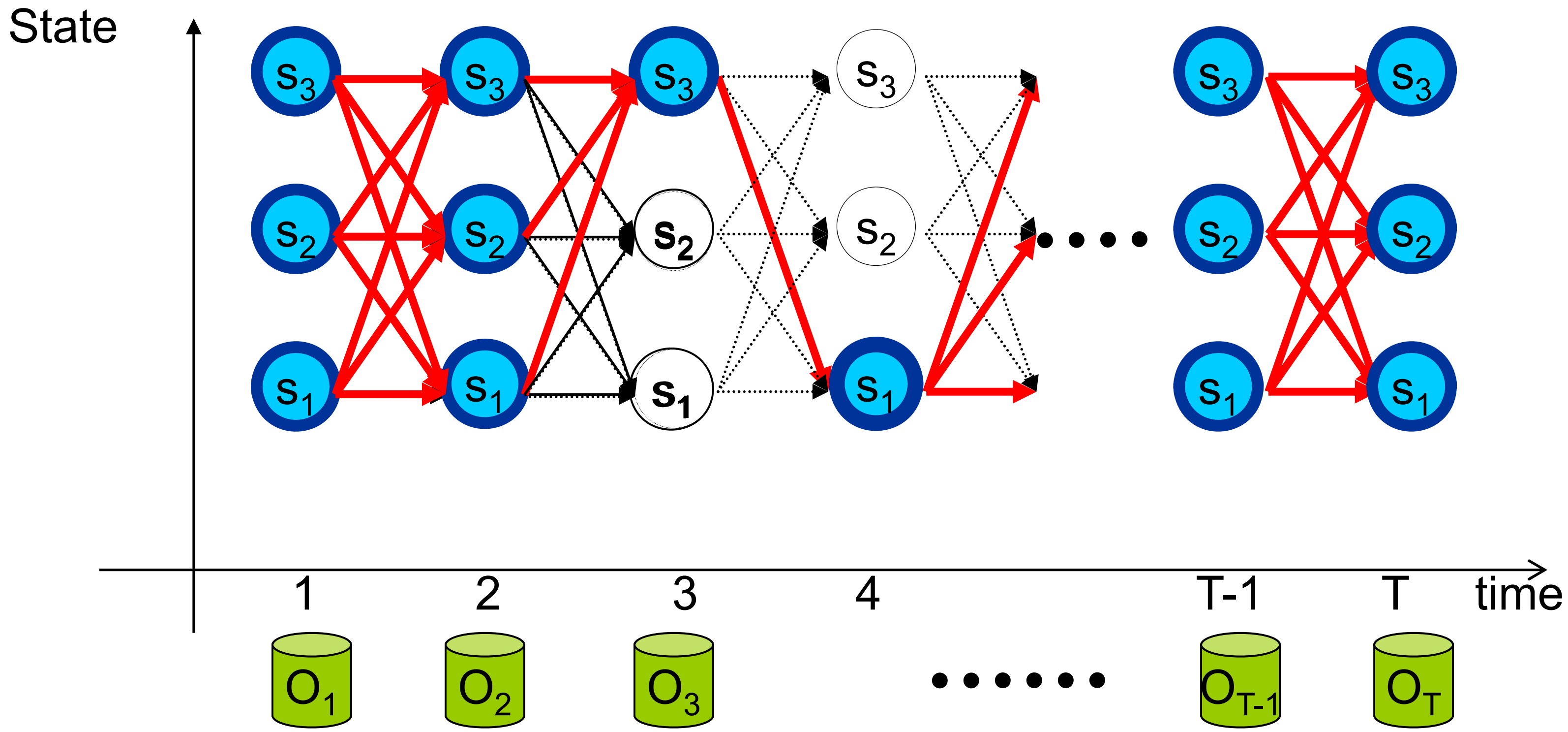
$$\gamma_t(i) = \sum_{j=1}^{N} P(s_t = i, s_{t+1} = j | \boldsymbol{O}, \lambda) = \sum_{j=1}^{N} \xi_t(i,j) \quad (\text{for } t < T)$$

Basic Problem 3 of HMM

Intuitive View (cont.)



- $P(s_3 = 3, s_4 = 1, \mathbf{O} \mid \lambda) = \alpha_3(3) * a_{31} * b_1(o_4) * \beta_1(4)$



Basic Problem 3 of HMM
Intuitive View (cont.)

- $\xi_t(i,j) = P(s_t = i, s_{t+1} = j | \boldsymbol{O}, \boldsymbol{\lambda})$

$$\sum_{t=1}^{T-1} \xi_t(i,j) = \text{expected number of transitions from state } i \text{ to state } j \text{ in } \boldsymbol{O}$$

- $\gamma_t(i) = P(s_t = i | \boldsymbol{O}, \lambda)$

$$\sum_{t=1}^{T-1} \gamma_t(i) = \sum_{t=1}^{T-1} \sum_{j=1}^{N} \xi_t(i,j) = \text{expected number of transitions from state } i \text{ in } \boldsymbol{O}$$

- A set of reasonable re-estimation formula for {***A***, π} is

$$\overline{\pi}_i = \text{expected freqency (number of times) in state } i \text{ at time } t = 1$$
$$= \gamma_1(i)$$

$$\overline{a}_{ij} = \frac{\text{expected number of transitio n from state } i \text{ to state } j}{\text{expected number of transitio n from state } i} = \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{t=1}^{T-1} \gamma_t(i)}$$

Formulae for Single Training Utterance

Basic Problem 3 of HMM
Intuitive View (cont.)

- A set of reasonable re-estimation formula for $\{B\}$ is
  - For discrete and finite observation $b_j(\mathbf{v}_k)=P(\mathbf{o}_t=\mathbf{v}_k|\mathbf{s}_t=j)$

$$\bar{b}_j(\boldsymbol{v}_k)=\bar{P}(\boldsymbol{o}=\boldsymbol{v}_k|s=j)=\frac{\text{expected number of times in state } j \text{ and observing symbol } \boldsymbol{v}_k}{\text{expected number of times in state } j}=\frac{\sum\limits_{\substack{\mathrm{t=1}\\ \text{such that } \boldsymbol{o}=\boldsymbol{v}_k}}^{\mathrm{T}}\gamma_t(j)}{\sum\limits_{\mathrm{t=1}}^{\mathrm{T}}\gamma_t(j)}$$

  - For continuous and infinite observation $b_j(\mathbf{v})=f_{\mathbf{O}|\mathbf{S}}(\mathbf{o}_t=\mathbf{v}|\mathbf{s}_t=j)$,

$$\bar{b}_j(\boldsymbol{v})=\sum_{k=1}^{M}\bar{c}_{jk}N(\boldsymbol{v};\bar{\boldsymbol{\mu}}_{jk},\bar{\boldsymbol{\Sigma}}_{jk})=\sum_{k=1}^{M}\bar{c}_{jk}\left(\frac{1}{\left(\sqrt{2\pi}\right)^{L}\left|\bar{\boldsymbol{\Sigma}}_{jk}\right|^{1/2}}\exp\left(-\frac{1}{2}(\boldsymbol{v}-\bar{\boldsymbol{\mu}}_{jk})^{t}\bar{\Sigma}_{jk}^{-1}(\boldsymbol{v}-\bar{\boldsymbol{\mu}}_{jk})\right)\right)$$

Modeled as a mixture of multivariate Gaussian distributions

Basic Problem 3 of HMM
Intuitive View (cont.)

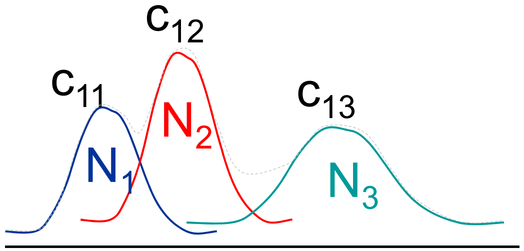
$$p(A|B)=\frac{p(A,B)}{P(B)}$$

- For continuous and infinite observation (Cont.)
  - Define a new variable $\gamma_t(j,k)$
    - $\gamma_t(j,k)$ is the probability of being in state *j* at time *t* with the *k*-th mixture component accounting for $\boldsymbol{o}_t$

$$\begin{aligned}
\gamma_t(j,k) &= P(s_t=j, m_t=k|\mathbf{O},\boldsymbol{\lambda})\\
&= P(s_t=j|\mathbf{O},\boldsymbol{\lambda})P(m_t=k|s_t=j,\mathbf{O},\boldsymbol{\lambda})\\
&= \gamma_t(j)P(m_t=k|s_t=j,\mathbf{O},\boldsymbol{\lambda})\\
&= \gamma_t(j)\frac{p(m_t=k,\mathbf{O}|s_t=j,\boldsymbol{\lambda})}{p(\mathbf{O}|s_t=j,\boldsymbol{\lambda})}\\
&= \gamma_t(j)\frac{P(m_t=k|s_t=j,\boldsymbol{\lambda})p(\mathbf{O}|s_t=j,m_t=k,\boldsymbol{\lambda})}{p(\mathbf{O}|s_t=j,\boldsymbol{\lambda})}
\end{aligned}$$

$$\text{......} \quad \text{(observation-independent assumption is applied)}$$

$$\begin{aligned}
&= \gamma_t(j)\frac{P(m_t=k|s_t=j,\boldsymbol{\lambda})p(\mathbf{o}_t|s_t=j,m_t=k,\boldsymbol{\lambda})}{p(\mathbf{o}_t|s_t=j,\boldsymbol{\lambda})}\\
&= \left[\frac{\alpha_t(j)\beta_t(j)}{\sum_{s=1}^{N}\alpha_t(s)\beta_t(s)}\right]\left[\frac{c_{jk}N(\mathbf{o}_t;\boldsymbol{\mu}_{jk},\boldsymbol{\Sigma}_{jk})}{\sum_{m=1}^{M}c_{jm}N(\mathbf{o}_t;\boldsymbol{\mu}_{jm},\boldsymbol{\Sigma}_{jm})}\right]
\end{aligned}$$



Distribution for State 1

$$\text{Note}: \gamma_t(j)=\sum_{m=1}^{M}\gamma_t(j,m)$$

Basic Problem 3 of HMM

Intuitive View (cont.)

– For continuous and infinite observation (Cont.)

$$\bar{c}_{jk} = \frac{\text{expected number of times in state } j \text{ and mixture } k}{\text{expected number of times in state } j} = \frac{\sum_{t=1}^{T} \gamma_t(j,k)}{\sum_{t=1}^{T} \sum_{m=1}^{M} \gamma_t(j,m)}$$

$$\bar{\boldsymbol{\mu}}_{jk} = \text{weighted average (mean) of observatio ns at state } j \text{ and mixture } k = \frac{\sum_{t=1}^{T} \gamma_t(j,k) \cdot \boldsymbol{o}_t}{\sum_{t=1}^{T} \gamma_t(j,k)}$$

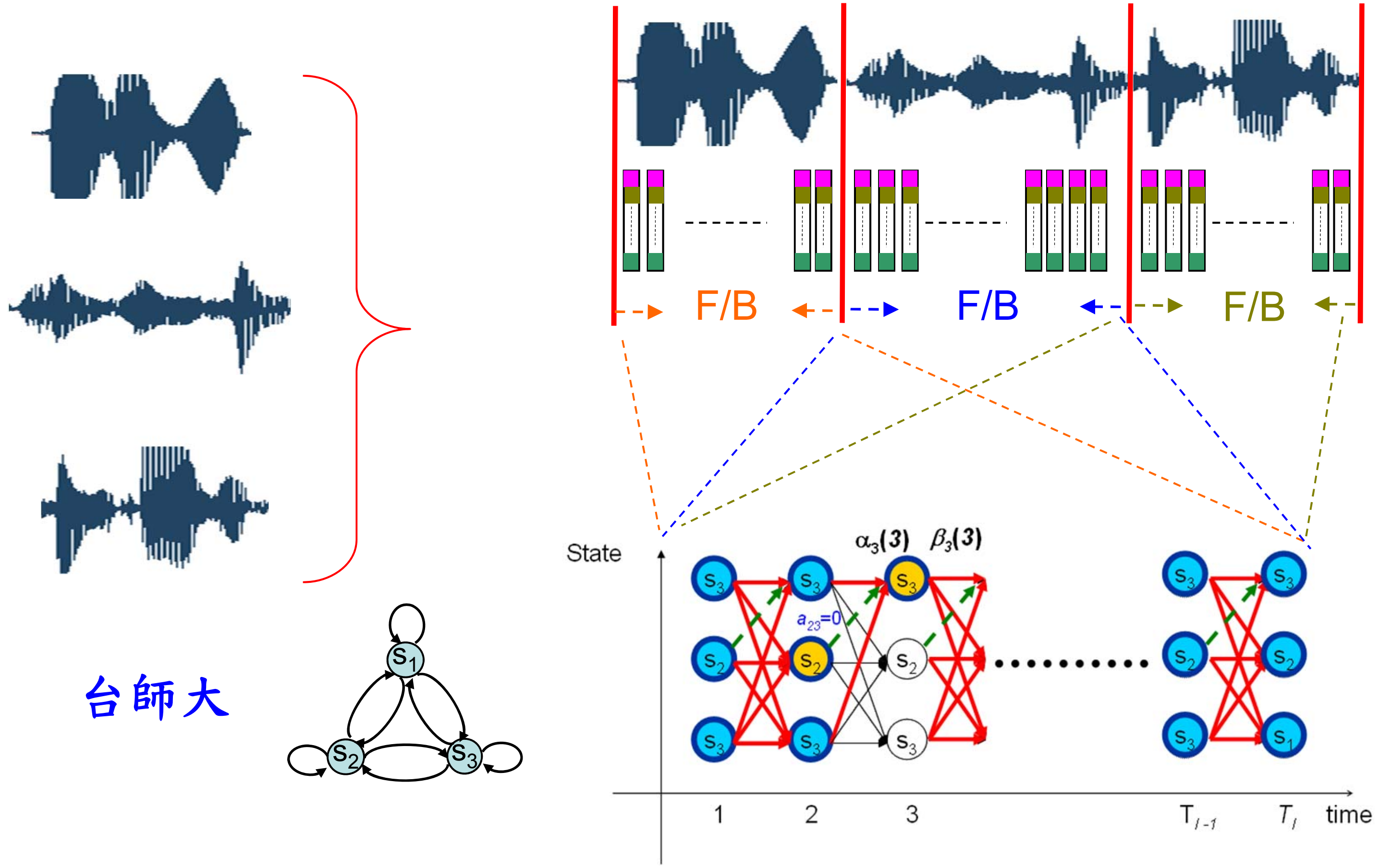
$$\bar{\boldsymbol{\Sigma}}_{jk} = \text{weighted covariance of observations at state } j \text{ and mixture } k$$

$$= \frac{\sum_{t=1}^{T} \gamma_t(j,k) \cdot \left(\boldsymbol{o}_t - \bar{\boldsymbol{\mu}}_{jk}\right)\left(\boldsymbol{o}_t - \bar{\boldsymbol{\mu}}_{jk}\right)^t}{\sum_{t=1}^{T} \gamma_t(j,k)}$$

Formulae for Single Training Utterance

Basic Problem 3 of HMM
Intuitive View (cont.)

- Multiple Training Utterances



Basic Problem 3 of HMM
Intuitive View (cont.)

– For continuous and infinite observation (Cont.)

$$\bar{\pi}_i = \text{expected freqency (number of times) in state } i \text{ at time } (t=1) = \frac{1}{L}\sum_{l=1}^{L}\gamma_1^l(i)$$

$$\bar{a}_{ij} = \frac{\text{expected number of transition from state } i \text{ to state } j}{\text{expected number of transition from state } i} = \frac{\sum_{l=1}^{L}\sum_{t=1}^{T_l\text{-}1}\xi_t^l(i,j)}{\sum_{l=1}^{L}\sum_{t=1}^{T_l\text{-}1}\gamma_t^l(i)}$$

$$\bar{c}_{jk} = \frac{\text{expected number of times in state } j \text{ and mixture } k}{\text{expected number of times in state } j} = \frac{\sum_{l=1}^{L}\sum_{t=1}^{T_l}\gamma_t^l(j,k)}{\sum_{l=1}^{L}\sum_{t=1}^{T_l}\sum_{m=1}^{M}\gamma_t^l(j,m)}$$

$$\bar{\boldsymbol{\mu}}_{jk} = \text{weighted average (mean) of observations at state } j \text{ and mixture } k = \frac{\sum_{l=1}^{L}\sum_{t=1}^{T_l}\gamma_t^l(j,k)\cdot\mathbf{o}_t}{\sum_{l=1}^{L}\sum_{t=1}^{T_l}\gamma_t^l(j,k)}$$

$$\bar{\boldsymbol{\Sigma}}_{jk} = \text{weighted covariance of observations at state } j \text{ and mixture } k$$

$$= \frac{\sum_{l=1}^{L}\sum_{t=1}^{T_l}\gamma_t^l(j,k)\cdot\left(\mathbf{o}_t-\bar{\boldsymbol{\mu}}_{jk}\right)\left(\mathbf{o}_t-\bar{\boldsymbol{\mu}}_{jk}\right)^t}{\sum_{l=1}^{L}\sum_{t=1}^{T_l}\gamma_t^l(j,k)}$$

Formulae for Multiple (L) Training Utterances

Basic Problem 3 of HMM
Intuitive View (cont.)

– For discrete and finite observation (cont.)

$$\bar{\pi}_i = \text{expected freqency (number of times) in state } i \text{ at time } (t=1) = \frac{1}{L}\sum_{l=1}^{L}\gamma_1^l(i)$$

$$\bar{a}_{ij} = \frac{\text{expected number of transition from state } i \text{ to state } j}{\text{expected number of transition from state } i} = \frac{\sum_{l=1}^{L}\sum_{t=1}^{T_l\text{-}1}\xi_t^l(i,j)}{\sum_{l=1}^{L}\sum_{t=1}^{T_l\text{-}1}\gamma_t^l(i)}$$

$$\bar{b}_j(\mathbf{v}_k) = \bar{P}(\mathbf{o}=\mathbf{v}_k|s=j) = \frac{\text{expected number of times in state } j \text{ and observing symbol } \mathbf{v}_k}{\text{expected number of times in state } j} = \frac{\sum_{l=1}^{L}\sum_{\substack{\text{t}=1\\ \text{such that } \mathbf{o}=\mathbf{v}_k}}^{T_l}\gamma_t^l(j)}{\sum_{l=1}^{L}\sum_{\text{t}=1}^{T_l}\gamma_t^l(j)}$$

Formulae for Multiple (L) Training Utterances

Semicontinuous HMMs

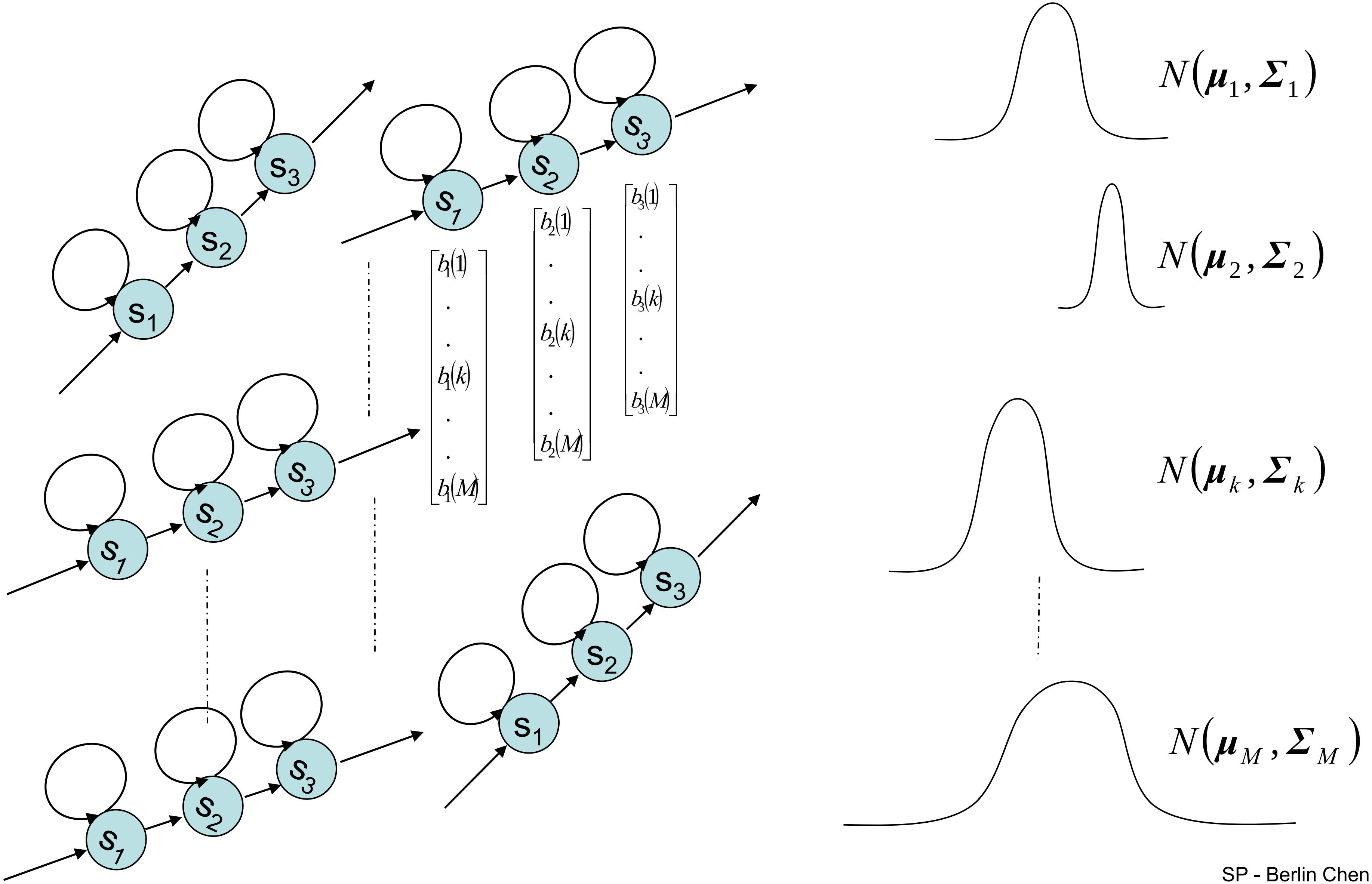
- The HMM state mixture density functions are tied together across all the models to form a set of shared kernels
  - The semicontinuous or tied-mixture HMM

$$b_j(\boldsymbol{o}) = \sum_{k=1}^{M} b_j(k) f(\boldsymbol{o}|v_k) = \sum_{k=1}^{M} \underbrace{b_j(k)}_{\substack{k\text{-th mixture weight} \\ \text{t of state } j \\ \text{(discrete, model-dependent)}}} \underbrace{N(\boldsymbol{o}, \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)}_{\substack{k\text{-th mixture density function or } k\text{-th codeword} \\ \text{(shared across HMMs, } M \text{ is very large)}}}$$

state output
Probability of state *j*

  - A combination of the discrete HMM and the continuous HMM
    - A combination of *discrete* model-dependent weight coefficients and *continuous* model-independent codebook probability density functions
  - Because *M* is large, we can simply use the *L* most significant values $f(\boldsymbol{o}|v_k)$
    - Experience showed that *L* is *1~3%* of *M* is adequate
  - Partial tying of $f(\boldsymbol{o}|v_k)$ for different phonetic class

Semicontinuous HMMs (cont.)



HMM Topology

- Speech is time-evolving non-stationary signal
  - Each HMM state has the ability to capture some quasi-stationary segment in the non-stationary speech signal
  - A *left-to-right* topology is a natural candidate to model the speech signal (also called the “beads-on-a-string” model)

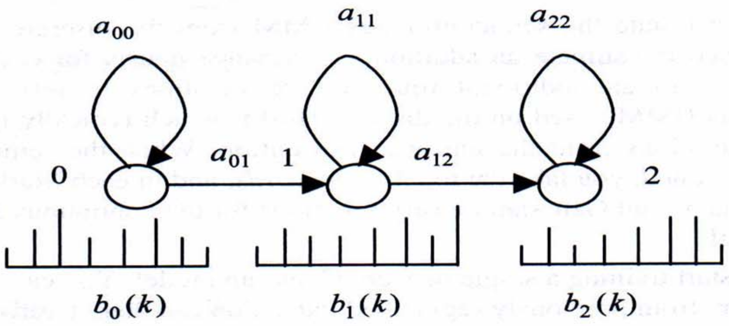
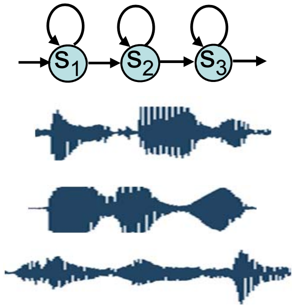


Figure 8.8 A typical hidden Markov model used to model phonemes. There are three states (0-2) and each state has an associated output probability distribution.

  - It is general to represent a phone using 3~5 states (English) and a syllable using 6~8 states (Mandarin Chinese)



Initialization of HMM

- A good initialization of HMM training :
  Segmental K-Means Segmentation into States
  - Assume that we have a training set of observations and an initial estimate of all model parameters
  - Step 1 : The set of training observation sequences is segmented into states, based on the initial model (finding the optimal state sequence by *Viterbi* Algorithm)
  - Step 2 :
    - For discrete density HMM (using M-codeword codebook)

$$\bar{b}_j(k) = \frac{\text{the number of vectors with codebook index } k \text{ in state } j}{\text{the number of vectors in state } j}$$

    - For continuous density HMM (M Gaussian mixtures per state)

$$\Rightarrow \text{cluster the observation vectors within each state } j \text{ into a set of } M \text{ clusters}$$

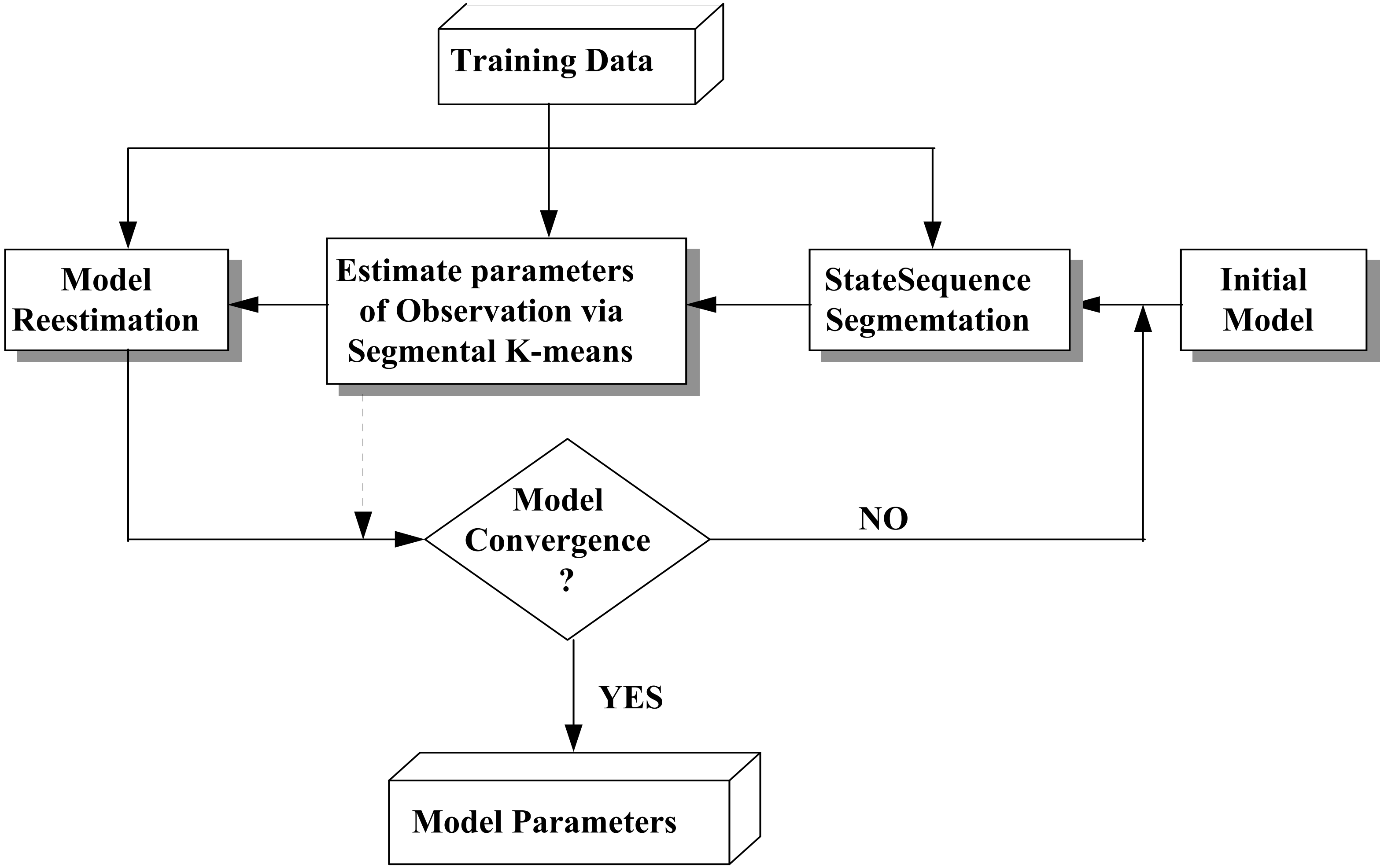
$$\bar{w}_{jm} = \text{number of vectors classified in cluster } m \text{ of state } j$$
$$\text{divided by the number of vectors in state } j$$

$$\bar{\mu}_{jm} = \text{sample mean of the vectors classified in cluster } m \text{ of state } j$$

$$\bar{\Sigma}_{jm} = \text{sample covariance matrix of the vectors classified in cluster } m \text{ of state } j$$

  - Step 3: Evaluate the model score
    If the difference between the previous and current model scores is greater than a threshold, go back to Step 1, otherwise stop, the initial model is generated

Initialization of HMM (cont.)



Training Data

Model Reestimation

Estimate parameters of Observation via Segmental K-means

StateSequence Segmemtation

Initial Model

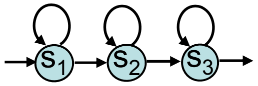
Model Convergence ?

NO

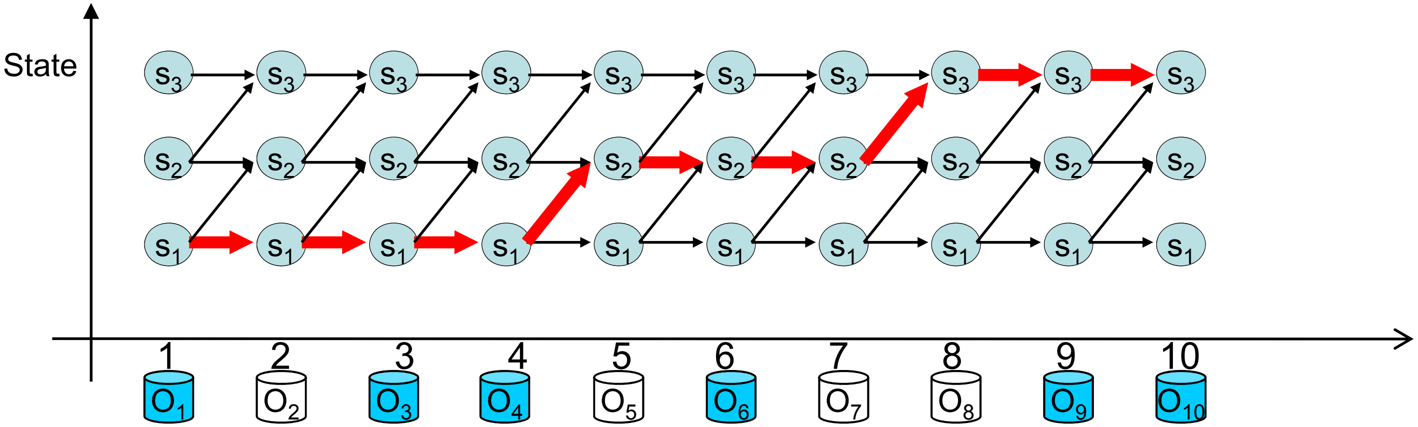
YES

Model Parameters

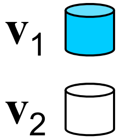
Initialization of HMM (cont.)



- An example for discrete HMM
  - 3 states and 2 codeword

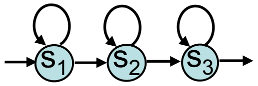


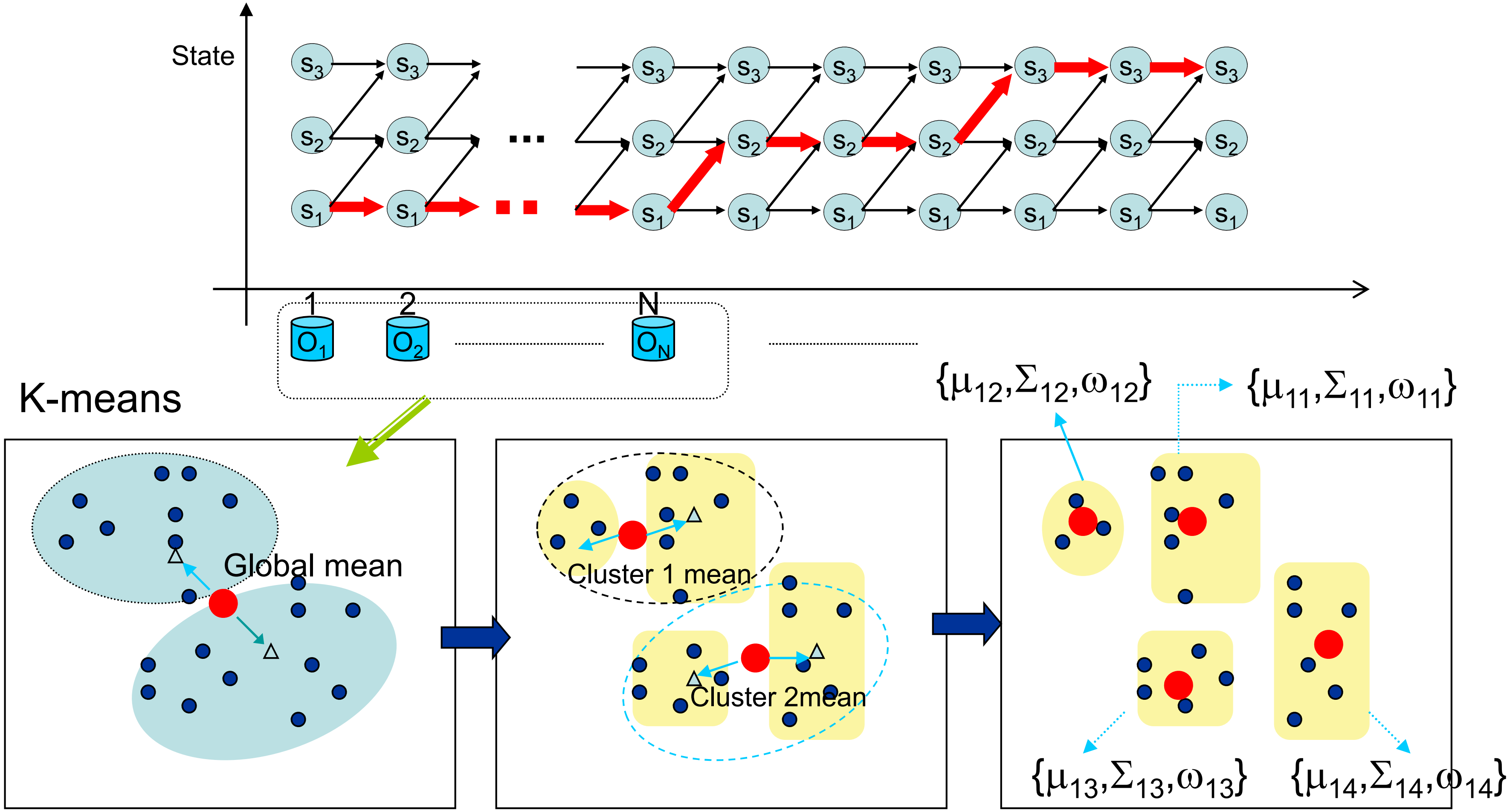
- $b_1(\mathbf{v}_1)=3/4$, $b_1(\mathbf{v}_2)=1/4$
- $b_2(\mathbf{v}_1)=1/3$, $b_2(\mathbf{v}_2)=2/3$
- $b_3(\mathbf{v}_1)=2/3$, $b_3(\mathbf{v}_2)=1/3$



Initialization of HMM (cont.)

- An example for Continuous HMM
  - 3 states and 4 Gaussian mixtures per state





Known Limitations of HMMs (1/3)

- The assumptions of conventional HMMs in Speech Processing
  - The state duration follows an exponential distribution
    - Don’t provide adequate representation of the temporal structure of speech

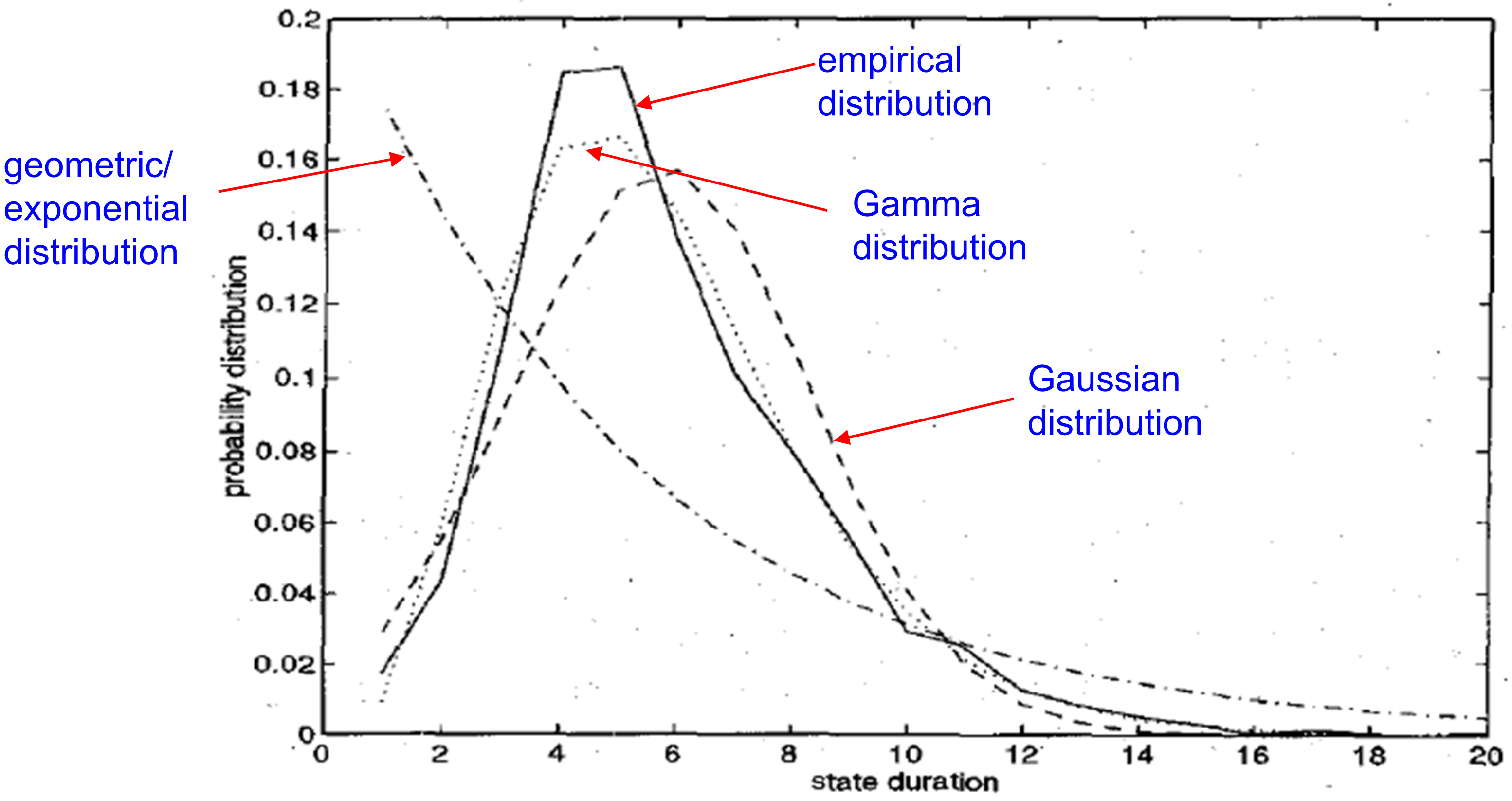
$$d_i(t) = a_{ii}^{t-1}(1 - a_{ii})$$

  - **First-order (Markov) assumption:** the state transition depends only on the origin and destination
  - **Output-independent assumption:** all observation frames are dependent on the state that generated them, not on neighboring observation frames

Researchers have proposed a number of techniques to address these limitations, albeit these solution have not significantly improved speech recognition accuracy for practical applications.

Known Limitations of HMMs (2/3)

- Duration modeling



Duration distributions for the seventh state of the word "seven:" empirical distribution (solid line); Gauss fit (dashed line); gamma fit (dotted line); and (d) geometric fit (dash-dot line).

Known Limitations of HMMs (3/3)

- The HMM parameters trained by the Baum-Welch algorithm (or EM algorithm) were only locally optimized