Hidden Markov Models for Speech Recognition

Berlin Chen 2003

References:

1. Rabiner and Juang, Fundamentals of Speech Recognition, Chapter 6
2. X. Huang et. al., Spoken Language Processing, Chapters 4, 8
3. Rabiner, “A Tutorial on Hidden Markov Models and Selected Applications in Speech Recognition,” Proceedings of the IEEE, vol. 77, No. 2, February 1989

Introduction

- Hidden Markov Model (HMM)

  History

  - Published in papers of Baum in late 1960s and early 1970s
  - Introduced to speech processing by Baker (CMU) and Jelinek (IBM) in the 1970s

  Assumption

  - Speech signal can be characterized as a parametric random process
  - Parameters can be estimated in a precise, well-defined manner

  Three fundamental problems

  - Evaluation of probability (likelihood) of a sequence of observations given a specific HMM
  - Determination of a best sequence of model states
  - Adjustment of model parameters so as to best account for observed signals

Observable Markov Model

- Observable Markov Model (Markov Chain)
  - **First-order** Markov chain of N states is a triple $(\boldsymbol{S},\boldsymbol{A},\pi)$
    - ***S*** is a set of N states
    - ***A*** is the $N \times N$ matrix of transition probabilities between states
      $P(s_t=j|s_{t-1}=i, s_{t-2}=k, \ldots\ldots)=P(s_t=j|s_{t-1}=i)\equiv A_{ij}$
    - π is the vector of initial state probability
      $\pi_j = P(s_1=j)$
  - The output of the process is the set of states at each instant of time, when each state corresponds to an observable event
  - The output in any given state is not random (***deterministic!***)
  - Too simple to describe the speech signal characteristics

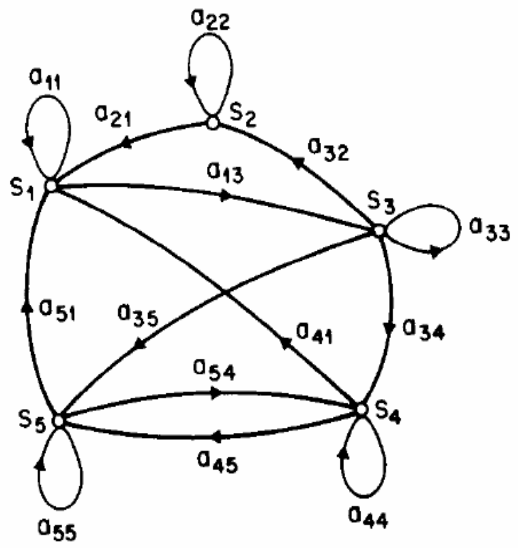
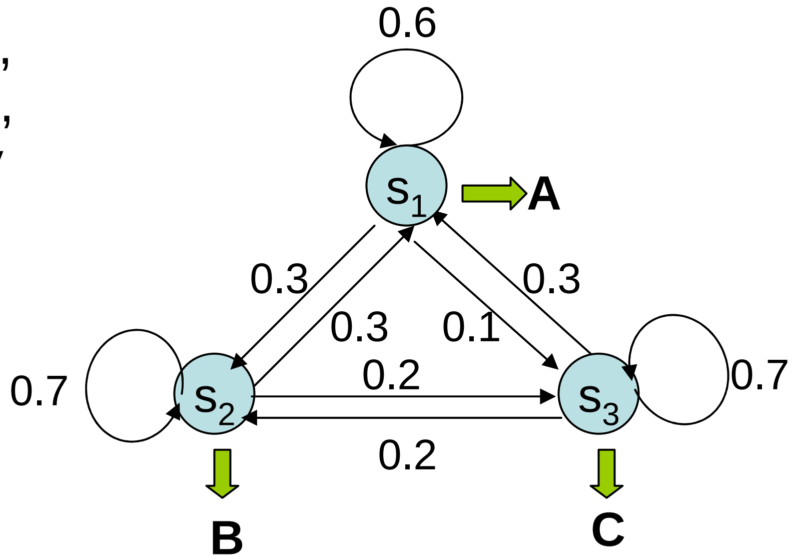


Fig. 1. A Markov chain with 5 states (labeled S_1 to S_5) with selected state transitions.

Observable Markov Model

- Example 1: A 3-state Markov Chain λ

State 1 generates symbol A **only**,
State 2 generates symbol B **only**,
and State 3 generates symbol C **only**



$$\mathbf{A} = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

$$\pi = \begin{bmatrix} 0.4 & 0.5 & 0.1 \end{bmatrix}$$

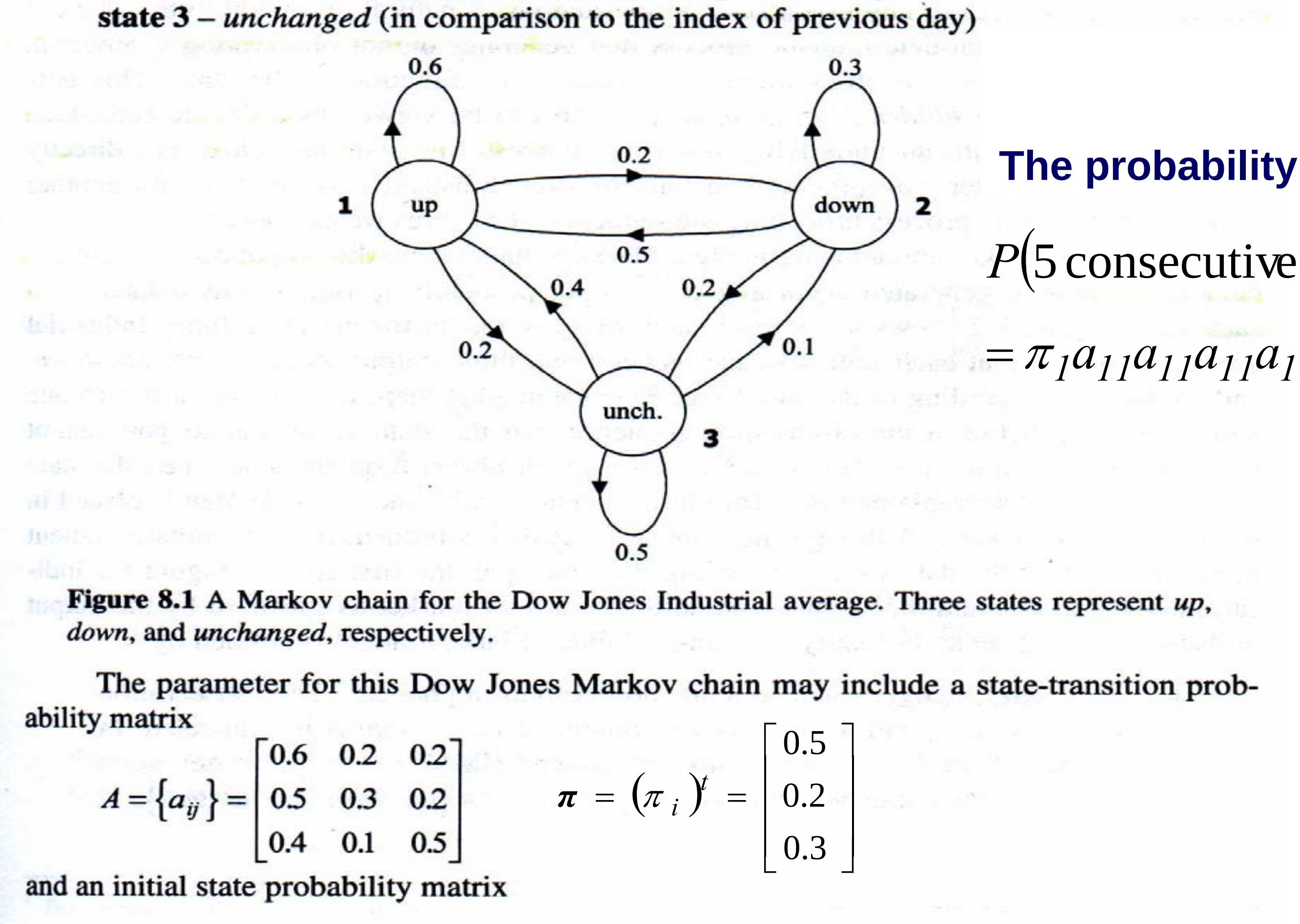
  - Given a sequence of observed symbols ***O***={CABBCABC}, the **only one** corresponding state sequence is {$S_3S_1S_2S_2S_3S_1S_2S_3$}, and the corresponding probability is

$P(\boldsymbol{O}|\lambda)$

$=P(S_3)P(S_1|S_3)P(S_2|S_1)P(S_2|S_2)P(S_3|S_2)P(S_1|S_3)P(S_2|S_1)P(S_3|S_2)$
$=0.1\times0.3\times0.3\times0.7\times0.2\times0.3\times0.3\times0.2=0.00002268$

Observable Markov Model

- Example 2: A three-state Markov chain for the *Dow Jones Industrial average*



state 1 – *up* (in comparison to the index of previous day)
state 2 – *down* (in comparison to the index of previous day)
state 3 – *unchanged* (in comparison to the index of previous day)

Figure 8.1 A Markov chain for the Dow Jones Industrial average. Three states represent *up*, *down*, and *unchanged*, respectively.

The parameter for this Dow Jones Markov chain may include a state-transition probability matrix

$$A = \{a_{ij}\} = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.5 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.5 \end{bmatrix} \qquad \boldsymbol{\pi} = (\pi_i)^t = \begin{bmatrix} 0.5 \\ 0.2 \\ 0.3 \end{bmatrix}$$

and an initial state probability matrix

The probability of 5 consecutive *up* days

$$P(5\ \text{consecutive}\ up\ \text{days}) = P(1,1,1,1,1)$$
$$= \pi_1 a_{11} a_{11} a_{11} a_{11} = 0.5 \times (0.6)^4 = 0.0648$$

Hidden Markov Model

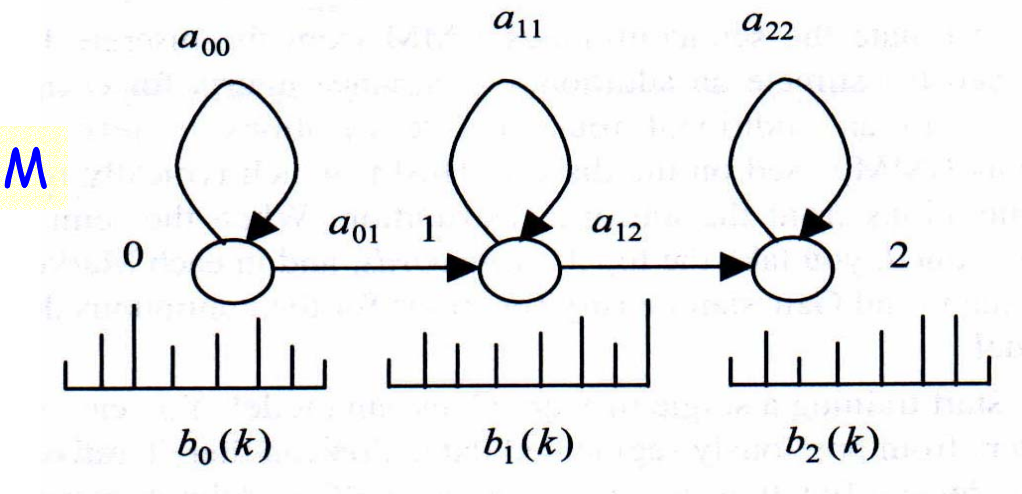
- HMM, an extended version of Observable Markov Model
  - The observation is turned to be **a probabilistic function (discrete or continuous) of a state** instead of an one-to-one correspondence of a state
  - The model is a doubly embedded stochastic process with an underlying stochastic process that is not directly observable (hidden)
    - What is hidden? ***The State Sequence!*** *According to the observation sequence, we are not sure which state sequence generates it!*
- Elements of an HMM (the State-Output HMM) $\lambda=\{\boldsymbol{S},\boldsymbol{A},\boldsymbol{B},\pi\}$
  - $\boldsymbol{S}$ is a set of N states
  - $\boldsymbol{A}$ is the $N \times N$ matrix of transition probabilities between states
  - $\boldsymbol{B}$ is a set of N probability functions, each describing the observation probability with respect to a state
  - π is the vector of initial state probability

Hidden Markov Model

- Two major assumptions
  - First order (Markov) assumption
    - The state transition depends only on the origin and destination
    - Time-invariant
  - Output-independent assumption
    - All observations are dependent on the state that generated them, not on neighboring observations

Hidden Markov Model

- Two major types of HMMs according to the observations
  - **Discrete and finite observations:**
    - The observations that all distinct states generate are finite in number
      $\boldsymbol{V}=\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \ldots\ldots, \boldsymbol{v}_M\}, \boldsymbol{v}_k \in \boldsymbol{R}^L$
    - In this case, the set of observation probability distributions $B=\{b_j(\boldsymbol{v}_k)\}$, is defined as $b_j(\boldsymbol{v}_k)=P(\mathbf{o}_t=\boldsymbol{v}_k|s_t=j)$, $1\leq k\leq M$, $1\leq j\leq N$
      $\mathbf{o}_t$: *observation at time t*, s_t : *state at time t*
      ⇨ *for state j*, $b_j(\boldsymbol{v}_k)$ consists of *only M probability values*



A left-to-right HMM

Hidden Markov Model

- Two major types of HMMs according to the observations
  - **Continuous and infinite observations:**
    - The observations that all distinct states generate are infinite and continuous, that is, $\boldsymbol{V}=\{\boldsymbol{v} | \boldsymbol{v} \in \boldsymbol{R}^L\}$
    - In this case, the set of observation probability distributions $B=\{b_j(\mathbf{v})\}$, is defined as $b_j(\mathbf{v})=f_{\mathbf{O}|\mathbf{S}}(\boldsymbol{o}_t=\mathbf{v}|s_t=j)$, $1 \le j \le N$
      ⇨ $b_j(\boldsymbol{v})$ is a ***continuous probability density function (pdf) and is often a mixture of Multivariate Gaussian (Normal) Distributions***

$$b_j(\boldsymbol{v}) = \sum_{k=1}^{M} w_{jk} \left(\frac{1}{(2\pi)^{L/2} \left| \boldsymbol{\Sigma}_{jk} \right|^{1/2}} \exp\left(-\frac{1}{2} \left(\boldsymbol{v} - \boldsymbol{\mu}_{jk} \right)^t \boldsymbol{\Sigma}_{jk}^{-1} \left(\boldsymbol{v} - \boldsymbol{\mu}_{jk} \right) \right) \right)$$

Mean Vector

Covariance Matrix

Observation Vector

Hidden Markov Model

- Multivariate Gaussian Distributions
  - When $\boldsymbol{X}$=($\boldsymbol{X}_1$, $\boldsymbol{X}_2$,…, $\boldsymbol{X}_n$) is a *n*-dimensional random vector, the multivariate Gaussian pdf has the form:

$$f\left(\boldsymbol{X}=\boldsymbol{x} \middle| \boldsymbol{\mu},\boldsymbol{\Sigma}\right)=N\left(\boldsymbol{x};\boldsymbol{\mu},\boldsymbol{\Sigma}\right)=\frac{1}{\left(2\pi\right)^{n/2}\left|\boldsymbol{\Sigma}\right|^{1/2}}\exp\left(-\frac{1}{2}\left(\boldsymbol{x}-\boldsymbol{\mu}\right)^{t}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{x}-\boldsymbol{\mu}\right)\right)$$

where $\boldsymbol{u}$ is the n-dimensional mean vector,

$\boldsymbol{\Sigma}$ is the coverance matrix, $\boldsymbol{\Sigma}=E\left[\left(\boldsymbol{x}-\boldsymbol{\mu}\right)\left(\boldsymbol{x}-\boldsymbol{\mu}\right)^{t}\right]=E\left[\boldsymbol{x}\boldsymbol{x}^{t}\right]-\boldsymbol{\mu}\boldsymbol{\mu}^{t}$

and $\left|\boldsymbol{\Sigma}\right|$ is the the dterminant of $\boldsymbol{\Sigma}$

The i-j^{th} elevment σ_{ij}^{2} of $\boldsymbol{\Sigma}$, $\sigma_{ij}^{2}=E\left[\left(x_i-\mu_i\right)\left(x_j-\mu_j\right)\right]=E\left[x_i x_j\right]-\mu_i\mu_j$

  - If $\boldsymbol{X}_1$, $\boldsymbol{X}_2$,…, $\boldsymbol{X}_n$ are independent, the covariance matrix is reduced to diagonal covariance
    - The distribution as ***n*** independent scalar Gaussian distributions
    - Model complexity is reduced

Hidden Markov Model

- Multivariate Gaussian Distributions

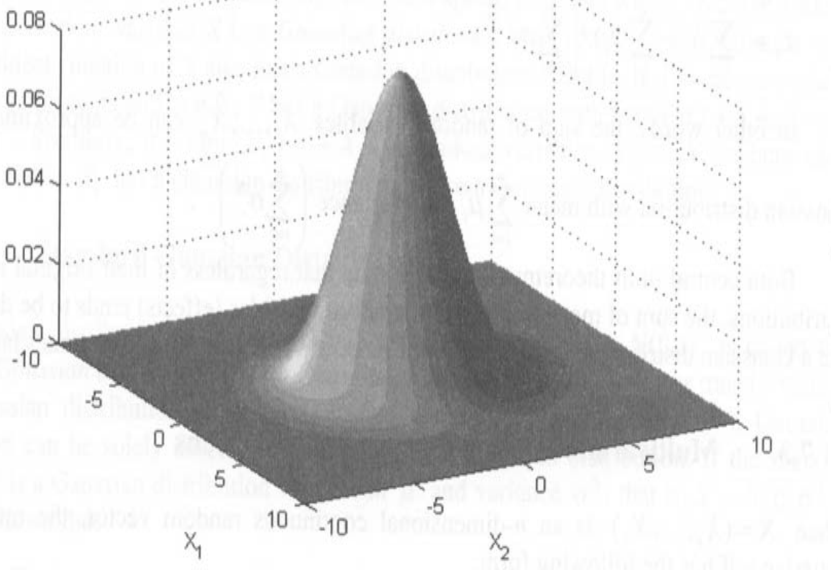


Figure 3.12 A two-dimensional multivariate Gaussian distribution with independent random variables x_1 and x_2 that have the same variance.

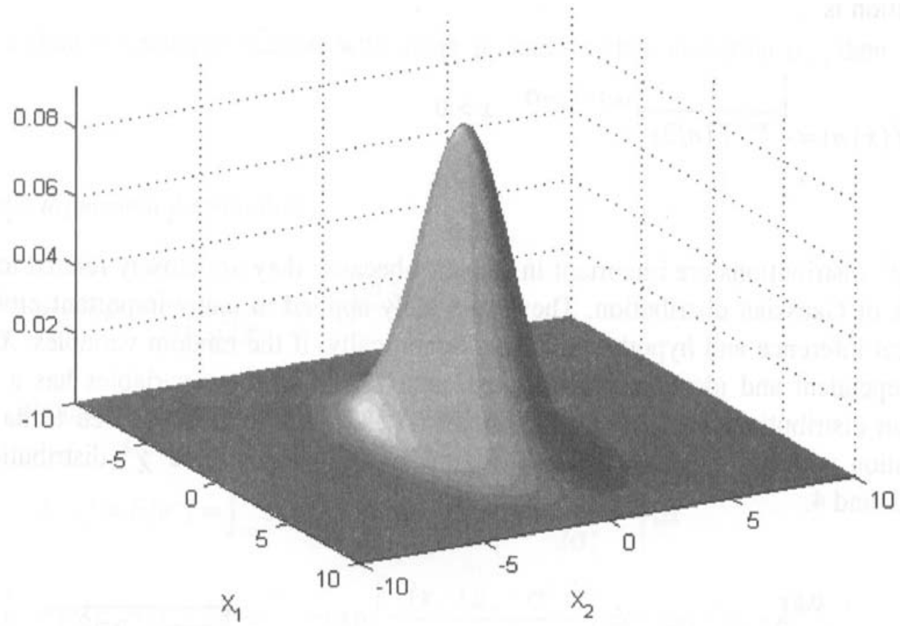
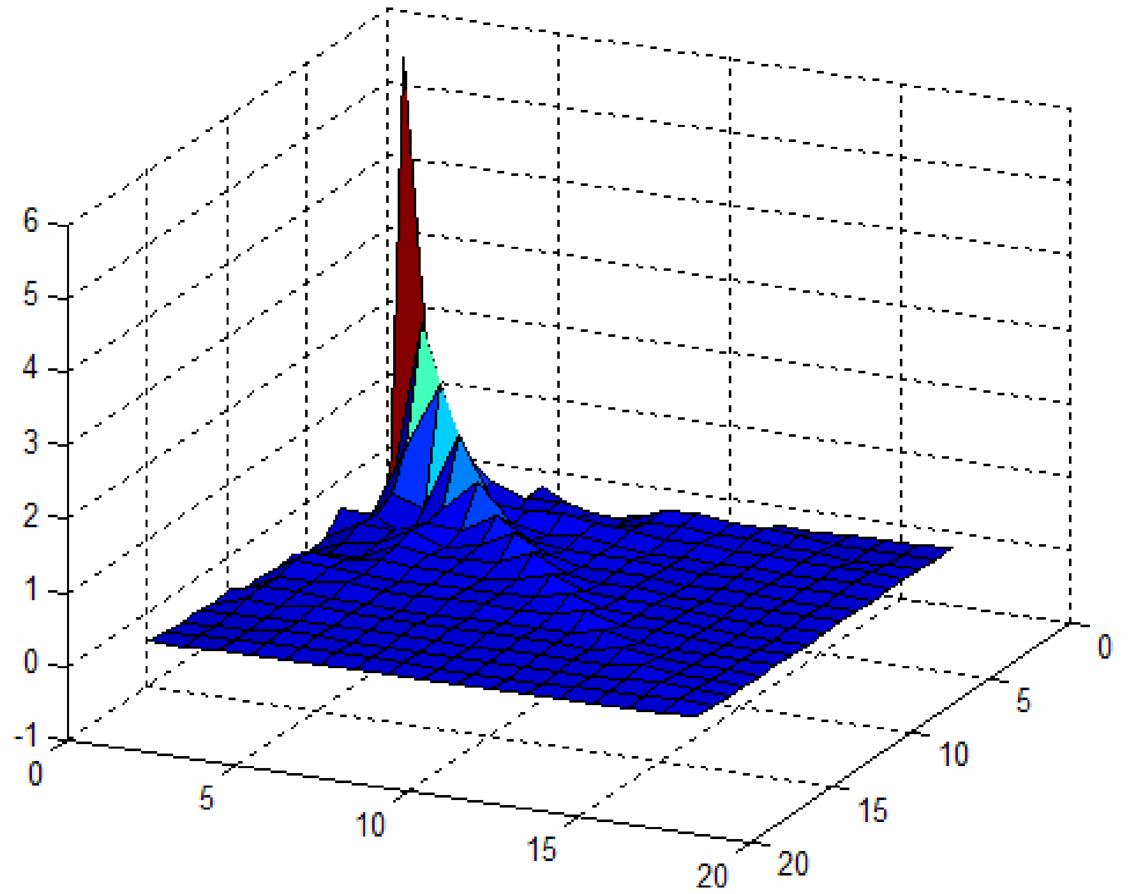


Figure 3.13 Another two-dimensional multivariate Gaussian distribution with independent random variable x_1 and x_2 which have different variances.

Hidden Markov Model

- Covariance matrix of the partially decorrelated feature vectors
  - MFCC cepstrum without C_0

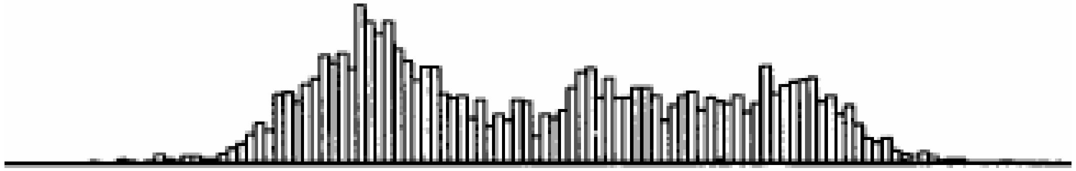


Hidden Markov Model

- Multivariate **Mixture** Gaussian Distributions (cont.)
  - More complex distributions with multiple local maxima can be approximated by Gaussian (a unimodal distribution) mixture

$$f(\boldsymbol{x}) = \sum_{k=1}^{M} w_k N_k(\boldsymbol{x}; \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k), \qquad \sum_{k=1}^{M} w_k = 1$$

  - Gaussian mixtures with enough mixture components can approximate any distribution



Hidden Markov Model

- Example : a 3-state discrete HMM λ

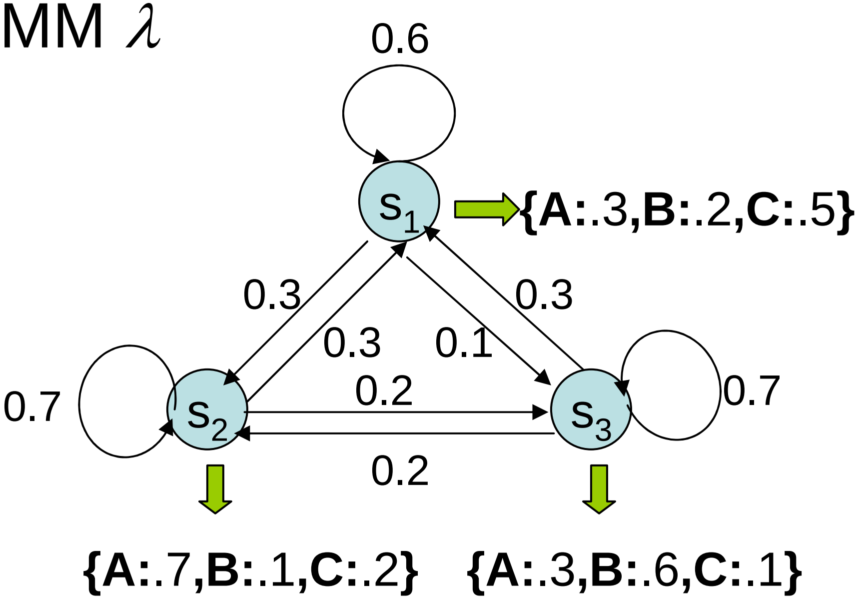
$$\mathbf{A} = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

$$b_1(\mathbf{A}) = 0.3, b_1(\mathbf{B}) = 0.2, b_1(\mathbf{C}) = 0.5$$

$$b_2(\mathbf{A}) = 0.7, b_2(\mathbf{B}) = 0.1, b_2(\mathbf{C}) = 0.2$$

$$b_3(\mathbf{A}) = 0.3, b_3(\mathbf{B}) = 0.6, b_3(\mathbf{C}) = 0.1$$

$$\pi = [0.4 \quad 0.5 \quad 0.1]$$



  - Given a sequence of observations *O={ABC}*, there are **27 possible** corresponding state sequences, and therefore the corresponding probability is

$$P(\boldsymbol{O}|\lambda) = \sum_{i=1}^{27} P(\boldsymbol{O}, \boldsymbol{S}_i|\lambda) = \sum_{i=1}^{27} P(\boldsymbol{O}|\boldsymbol{S}_i, \lambda) P(\boldsymbol{S}_i|\lambda), \qquad \boldsymbol{S}_i \text{ : state sequence}$$

$$E.g.\text{ when } \boldsymbol{S}_i = \{s_2 s_2 s_3\}, P(\boldsymbol{O}|\boldsymbol{S}_i, \boldsymbol{\lambda}) = P(\boldsymbol{A}|s_2) P(\boldsymbol{B}|s_2) P(\boldsymbol{C}|s_3) = 0.7 * 0.1 * 0.1 = 0.007$$

$$P(\boldsymbol{S}_i|\boldsymbol{\lambda}) = P(s_2) P(s_2|s_2) P(s_3|s_2) = 0.5 * 0.7 * 0.2 = 0.07$$

Hidden Markov Model

- Notation :
  - **$O=\{o_1o_2o_3\ldots\ldots o_T\}$:** the observation (feature) sequence
  - **$S=\{s_1s_2s_3\ldots\ldots s_T\}$:** the state sequence
  - λ : model, for HMM, $\lambda=\{\mathbf{A},\mathbf{B},\pi\}$
  - $P(O|\lambda)$: 用model λ 計算 ***O*** 的機率值
  - $P(O|S,\lambda)$:在***O***是state sequence ***S*** 所產生的前提下, 用model λ 計算 ***O*** 的機率值
  - $P(O,S|\lambda)$:用model λ 計算[***O*** ，***S***]兩者同時成立的機率值
  - $P(S|O,\lambda)$: 在已知***O***的前提下， 用model λ 計算 ***S*** 的機率值
- Useful formula
  - Bayesian Rule :

$$P(A|B)=\frac{P(A,B)}{P(B)}=\frac{P(B|A)P(A)}{P(B)} \quad\Longrightarrow\quad P(A|B,\boldsymbol{\lambda})=\frac{P(A,B|\boldsymbol{\lambda})}{P(B|\boldsymbol{\lambda})}=\frac{P(B|A,\boldsymbol{\lambda})P(A|\boldsymbol{\lambda})}{P(B|\boldsymbol{\lambda})}$$

$\boldsymbol{\lambda}$: model describing the probability

$$P(A,B)=P(B|A)P(A)=P(A|B)P(B)$$

Hidden Markov Model

- Useful formula (Cont.):

$$P(A)=\begin{cases}\sum_{all\ B} P(A,B)=\sum_{all\ B} P(A|B)P(B), & \text{if } B \text{ is disrete and disjoint}\\ \int_B f(A,B)dB=\int_B f(A|B)f(B)dB, & \text{if } B \text{ is continuous}\end{cases}$$

if $x_1,x_2,......,x_n$ are independent,

$$P(x_1,x_2,......,x_n)=P(x_1)P(x_2)......P(x_n)$$

$$E_z(q(z))=\begin{cases}\sum_k P(z=k)q(k), & z:\text{discrete}\\ \int_z f_z(z)q(z)dz, & z:\text{continuous}\end{cases}$$

Expectation

Hidden Markov Model

- Three Basic Problems for HMMs

  Given an observation sequence $O=(o_1,o_2,\ldots\ldots,o_T)$**,**

  and an HMM $\lambda=(S,A,B,\pi)$

  - **Problem** ***1***:

    How to *efficiently* compute $P(\mathbf{O}|\lambda)$?

    ⇨ ***Evaluation problem***

  - **Problem** ***2*****:**

    How to choose an optimal state sequence $\mathbf{S}=(s_1,s_2,\ldots\ldots, s_T)$?

    ⇨ ***Decoding Problem***

  - **Problem** ***3*****:**

    How to adjust the model parameter $\lambda=(A,B,\pi)$ to maximize $P(\mathbf{O}|\lambda)$?

    ⇨ ***Learning / Training Problem***

Basic Problem 1 of HMM

Given ***O*** and λ, find $P(\mathbf{O}|\lambda)$= Prob[observing *O* given λ]

- Direct Evaluation
  - Evaluating all possible state sequences of length *T* that generating observation sequence ***O***

$$P(\mathbf{O}|\lambda) = \sum_{all\ \mathbf{S}} P(\mathbf{O},\mathbf{S}|\lambda) = \sum_{all\ S} P(\mathbf{O}|\mathbf{S},\lambda)P(\mathbf{S}|\lambda)$$

  - $P(\mathbf{S}|\lambda)$: The probability of each path ***S***
    - By Markov assumption (First-order HMM)

$$P(\mathbf{S}|\lambda) = P(s_1|\lambda)\prod_{t=2}^{T} P\left(s_t\middle|s_1^{t-1},\lambda\right)$$

$$\approx P(s_1|\lambda)\prod_{t=2}^{T} P(s_t|s_{t-1},\lambda)$$

$$= \pi_{s_1} a_{s_1 s_2} a_{s_2 s_3} \ldots a_{s_{T-1} s_T}$$

Basic Problem 1 of HMM

- Direct Evaluation (cont.)
  - $P(\boldsymbol{O}|\boldsymbol{S},\lambda)$: The joint output probability along the path *S*
    - By output-independent assumption
      - The probability that a particular observation symbol/vector is emitted at time *t* depends only on the state s_t and is conditionally independent of the past observations

$$
\begin{aligned}
P(\boldsymbol{O}|\boldsymbol{S},\lambda) &= P\left(\boldsymbol{o}_1^T \middle| s_1^T, \lambda\right) \\
&= P\left(\boldsymbol{o}_1 \middle| s_1^T, \lambda\right) \prod_{t=2}^{T} P\left(\boldsymbol{o}_t \middle| \boldsymbol{o}_1^{t-1}, s_1^T, \lambda\right) \\
&\approx \prod_{t=1}^{T} P\left(\boldsymbol{o}_t \middle| s_t, \lambda\right) \\
&= \prod_{t=1}^{T} b_{s_t}(\boldsymbol{o}_t)
\end{aligned}
$$

Basic Problem 1 of HMM

- Direct Evaluation (Cont.)

$$P(\boldsymbol{O}|\lambda)=\sum_{all\ \boldsymbol{S}} P(\boldsymbol{S}|\lambda)P(\boldsymbol{O}|\boldsymbol{S},\lambda)$$

$$=\sum_{all\ \boldsymbol{s}}\left(\left[\pi_{s_1}a_{s_1s_2}a_{s_2s_3}\ldots\ldots a_{s_{T-1}s_T}\right]\left[b_{s_1}(\boldsymbol{o}_1)b_{s_2}(\boldsymbol{o}_2)\ldots\ldots b_{s_T}(\boldsymbol{o}_T)\right]\right)$$

$$=\sum_{s_1,s_2,..,s_T}\pi_{s_1}b_{s_1}(\boldsymbol{o}_1)a_{s_1s_2}b_{s_2}(\boldsymbol{o}_2)\ldots\ldots a_{s_{T-1}s_T}b_{s_T}(\boldsymbol{o}_T)$$

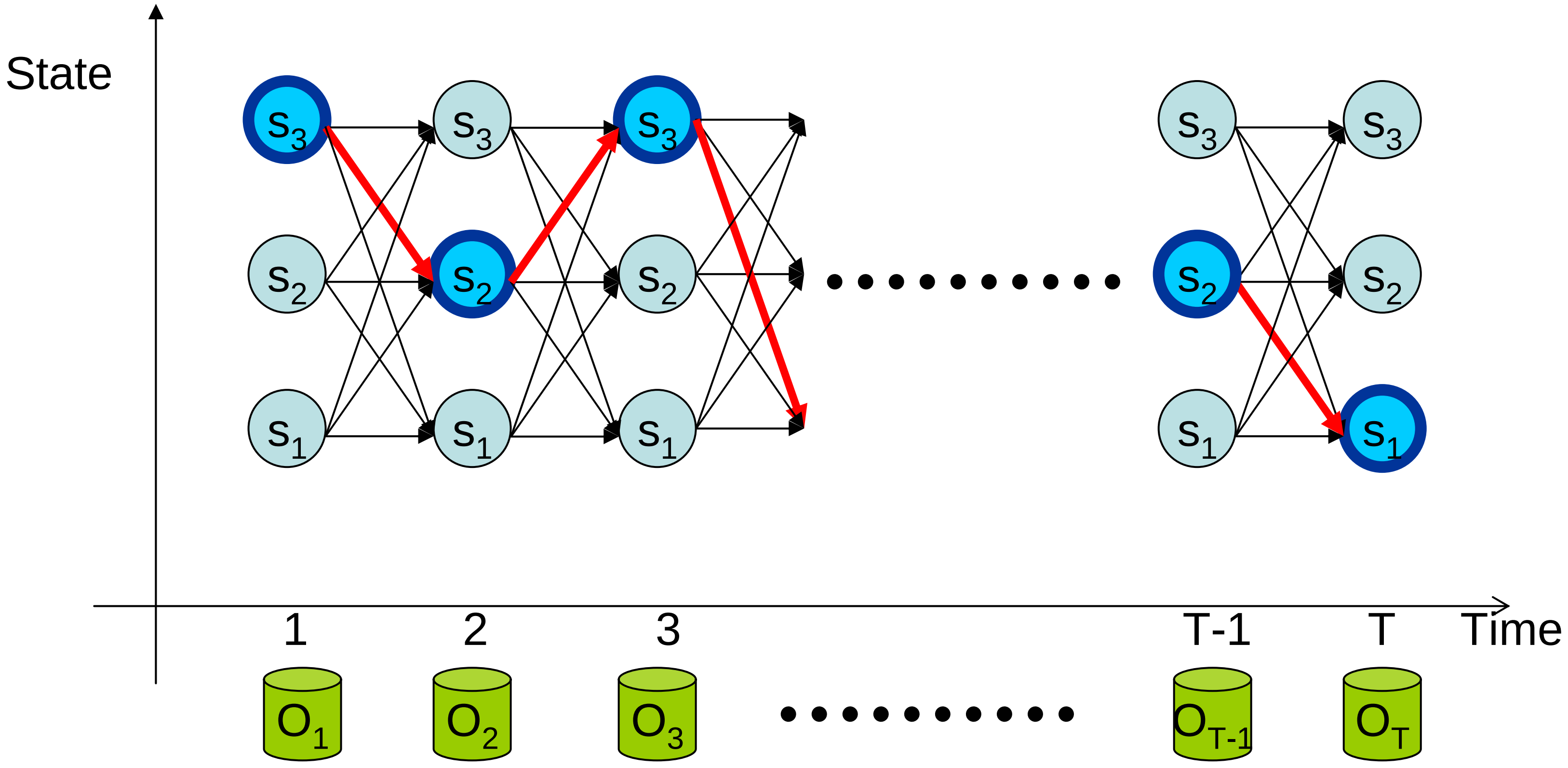
  - Huge Computation Requirements: $O(N^T)$
    - Exponential computational complexity

$$\textbf{\textit{Complexity}} \quad : (2T\text{-}1)N^{T}\ MUL \quad \approx 2TN^{T},\ N^{T}\text{-}1\ \text{ADD}$$

- A more efficient algorithms can be used to evaluate $P(\boldsymbol{O}|\lambda)$
  - *Forward/Backward Procedure/Algorithm*

Basic Problem 1 of HMM

- Direct Evaluation (Cont.)



s_i **means $b_j(o_t)$ has been computed**

a_{ij} → **means a_{ij} has been computed**

Basic Problem 1 of HMM

- The Forward Procedure

- Base on the HMM assumptions, the calculation of $P\left(s_t \middle| s_{t-1}, \lambda\right)$ and $P\left(\boldsymbol{o}_t \middle| s_t, \lambda\right)$ involves only s_{t-1}, s_t and $\boldsymbol{o}_t$, so it is possible to compute the likelihood with recursion on t

- Forward variable : $\alpha_t(i) = P\left(o_1 o_2 \ldots o_t, s_t = i \middle| \boldsymbol{\lambda}\right)$
  - The probability that the HMM is in state *i* at time *t* having generating partial observation $\boldsymbol{o}_1\boldsymbol{o}_2\ldots\boldsymbol{o}_t$

Basic Problem 1 of HMM

- The Forward Procedure

- Algorithm

1. Initialization $\alpha_1(i)=\pi_i b_i(\boldsymbol{o}_1),\ 1\le i\le N$

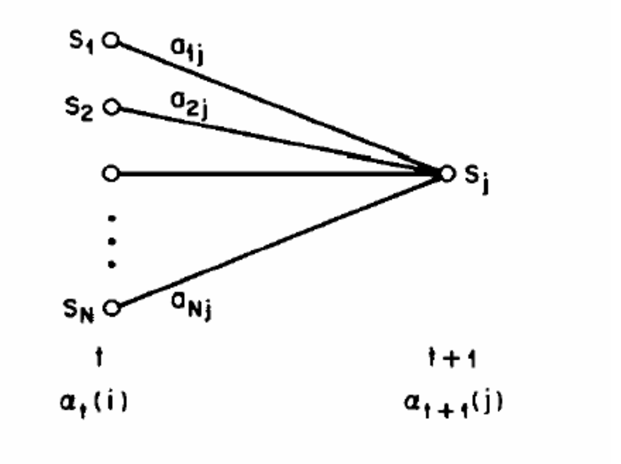
2. Induction $\alpha_{t+1}(j)=\left[\sum_{i=1}^{N}\alpha_t(i)a_{ij}\right]b_j(\boldsymbol{o}_{t+1}),\ \ 1\le t\le T\text{-}1, 1\le j\le N$

3. Termination $P(\boldsymbol{O}|\lambda)=\sum_{i=1}^{N}\alpha_T(i)$

  - Complexity: $O(N^2T)$

    $\text{MUL}\ :\ N(N+1)(T-1)+N \approx N^2T$

    $\text{ADD}\ :\ (N-1)N(T-1)+(N-1)\approx N^2T$



- Based on the lattice (trellis) structure
  - Computed in a *time-synchronous* fashion from *left-to-right*, where each cell for time *t* is completely computed before proceeding to time *t+1*
- All state sequences, regardless how long previously, merge to *N* nodes (states) at each time instance *t*

Basic Problem 1 of HMM

- The Forward Procedure

$$\alpha_t(j) = P(o_1 o_2 \ldots o_t, s_t = j | \lambda)$$

$$= P(o_1 o_2 \ldots o_t | s_t = j, \lambda) P(s_t = j | \lambda)$$

$$= P(o_1 o_2 \ldots o_{t-1} | s_t = j, \lambda) P(o_t | s_t = j, \lambda) P(s_t = j | \lambda)$$

$$= P(o_1 o_2 \ldots o_{t-1}, s_t = j | \lambda) P(o_t | s_t = j, \lambda)$$

$$= P(o_1 o_2 \ldots o_{t-1}, s_t = j | \lambda) b_j(o_t)$$

$$= \left[\sum_{i=1}^{N} P(o_1 o_2 \ldots o_{t-1}, s_{t-1} = i, s_t = j | \lambda) \right] b_j(o_t)$$

$$= \left[\sum_{i=1}^{N} P(o_1 o_2 \ldots o_{t-1}, s_{t-1} = i | \lambda) P(s_t = j | o_1 o_2 \ldots o_{t-1}, s_{t-1} = i, \lambda) \right] b_j(o_t)$$

$$= \left[\sum_{i=1}^{N} P(o_1 o_2 \ldots o_{t-1}, s_{t-1} = i | \lambda) P(s_t = j | s_{t-1} = i, \lambda) \right] b_j(o_t)$$

$$= \left[\sum_{i=1}^{N} \alpha_{t-1}(i) a_{ij} \right] b_j(o_t)$$

$P(A,B) = P(B|A)P(A)$

output independent assumption

$P(B|A)P(A) = P(A,B)$

$P(o_t | s_t = j, \lambda) = b_j(o_t)$

$P(A) \sum_{all\ B} P(A,B)$

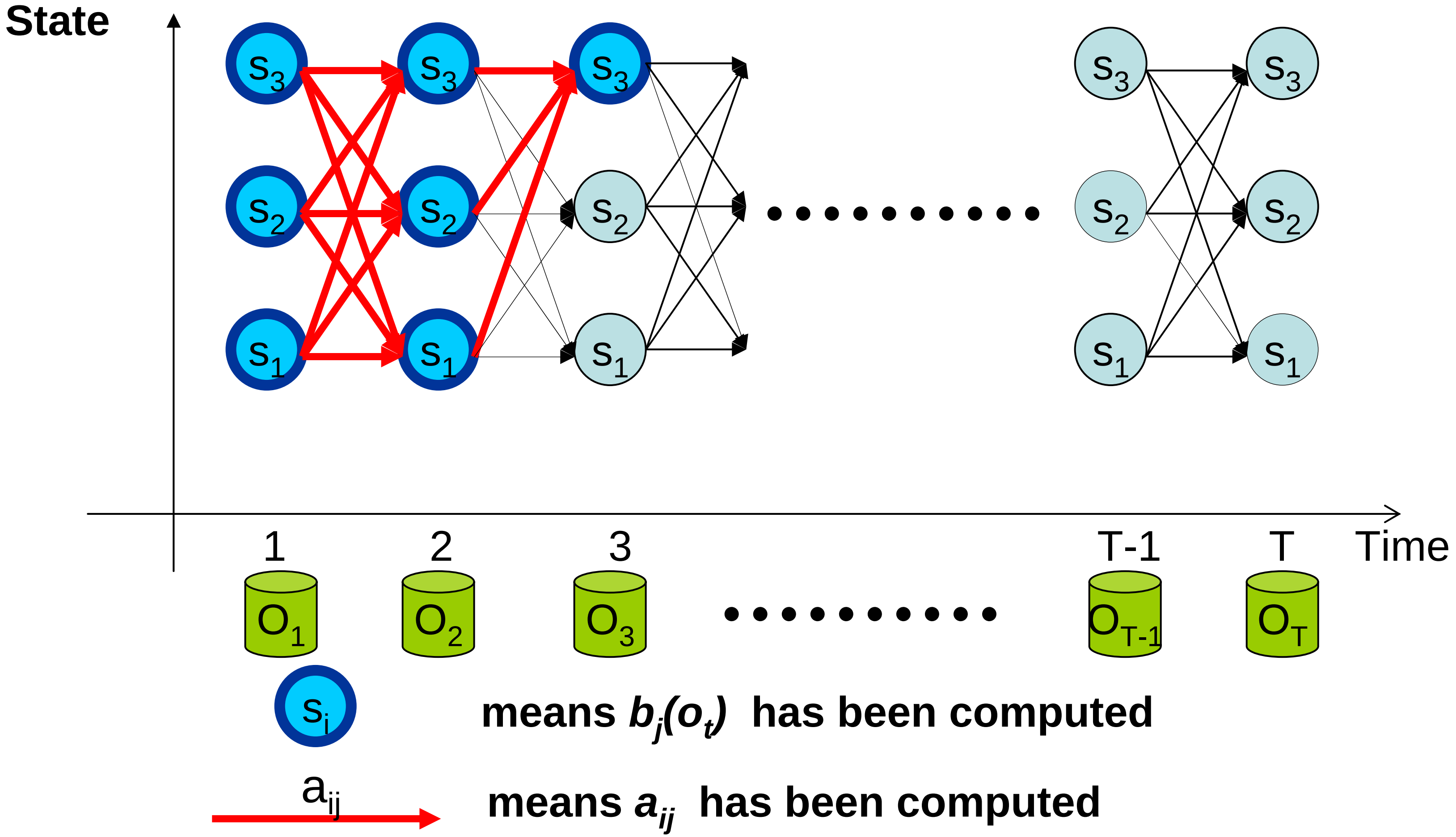
first-order Markov assumption

Basic Problem 1 of HMM

- The Forward Procedure

- $\alpha_3(3)=P(\mathbf{o}_1,\mathbf{o}_2,\mathbf{o}_3,s_3=3|\lambda)$

  $=[\alpha_2(1)*a_{13}+\alpha_2(2)*a_{23}+\alpha_2(3)*a_{33}]b_3(\mathbf{o}_3)$



means $b_j(o_t)$ has been computed

means a_{ij} has been computed

Basic Problem 1 of HMM

- The Forward Procedure

- A three-state Hidden Markov Model for the *Dow Jones Industrial average*

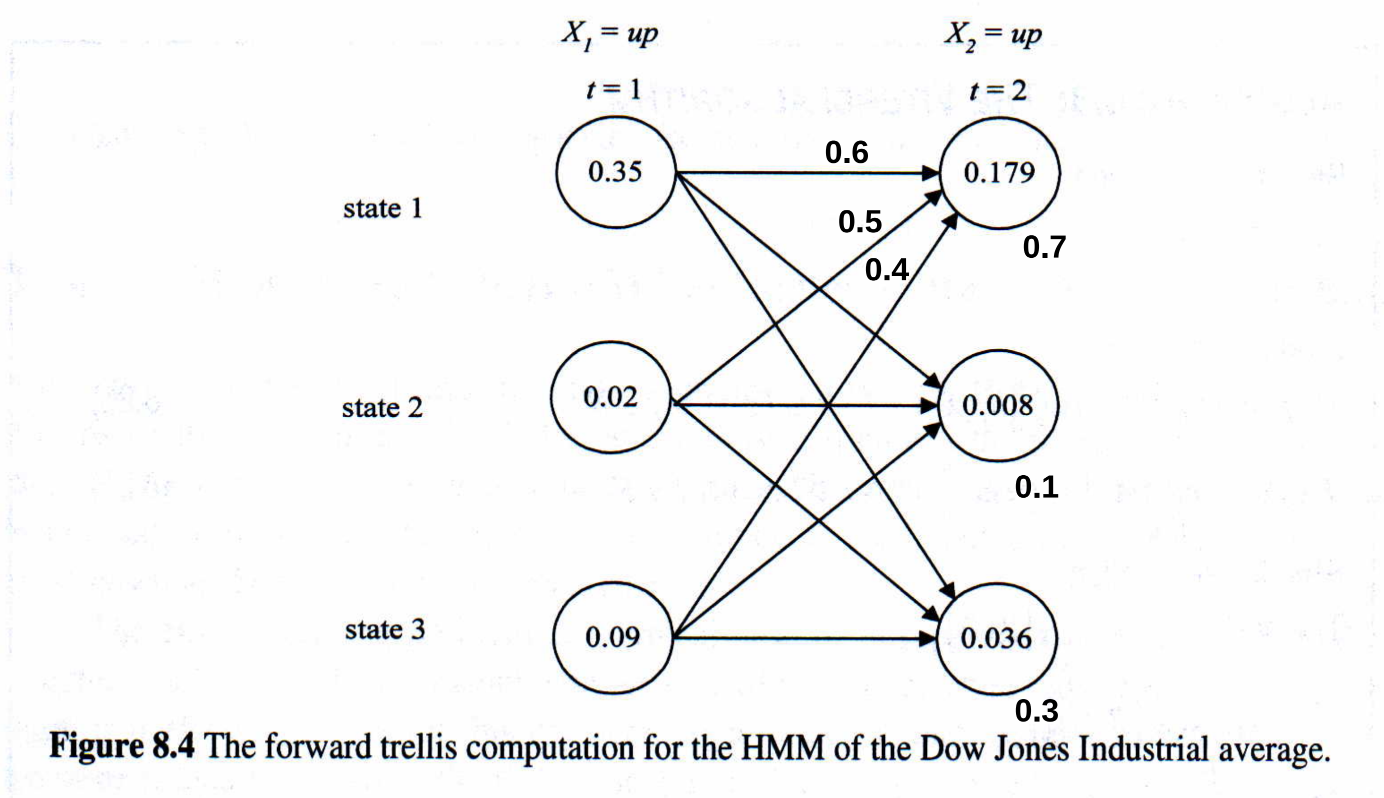


Figure 8.4 The forward trellis computation for the HMM of the Dow Jones Industrial average.

Basic Problem 1 of HMM

- The Backward Procedure

- Backward variable : $\beta_t(i)=P(\boldsymbol{o}_{t+1},\boldsymbol{o}_{t+2},\ldots..,\boldsymbol{o}_T|s_t=i\ ,\ \lambda)$

$$1.\ \text{Initialization}: \beta_{\mathrm{T}}(i)=1,\ 1\le i\le N$$

$$2.\ \text{Induction}: \beta_t(i)=\sum_{j=1}^{N}a_{ij}b_j(\boldsymbol{o}_{t+1})\beta_{t+1}(j),\ 1\le t\le T\text{-}1, 1\le j\le N$$

$$3.\ \text{Termination}: P(\boldsymbol{O}|\lambda)=\sum_{j=1}^{N}\pi_j b_j(\boldsymbol{o}_1)\beta_1(j)$$

$$\text{Complexity MUL}: 2N^2(T\text{-}1)+2N\approx\ N^2T\ ;$$

$$\text{ADD}: (N\text{-}1)N(T\text{-}1)+N\approx N^2T$$

Basic Problem 2 of HMM

- Why $P(\boldsymbol{O}, s_t = i|\lambda) = \alpha_t(i)\,\beta_t(i)$?

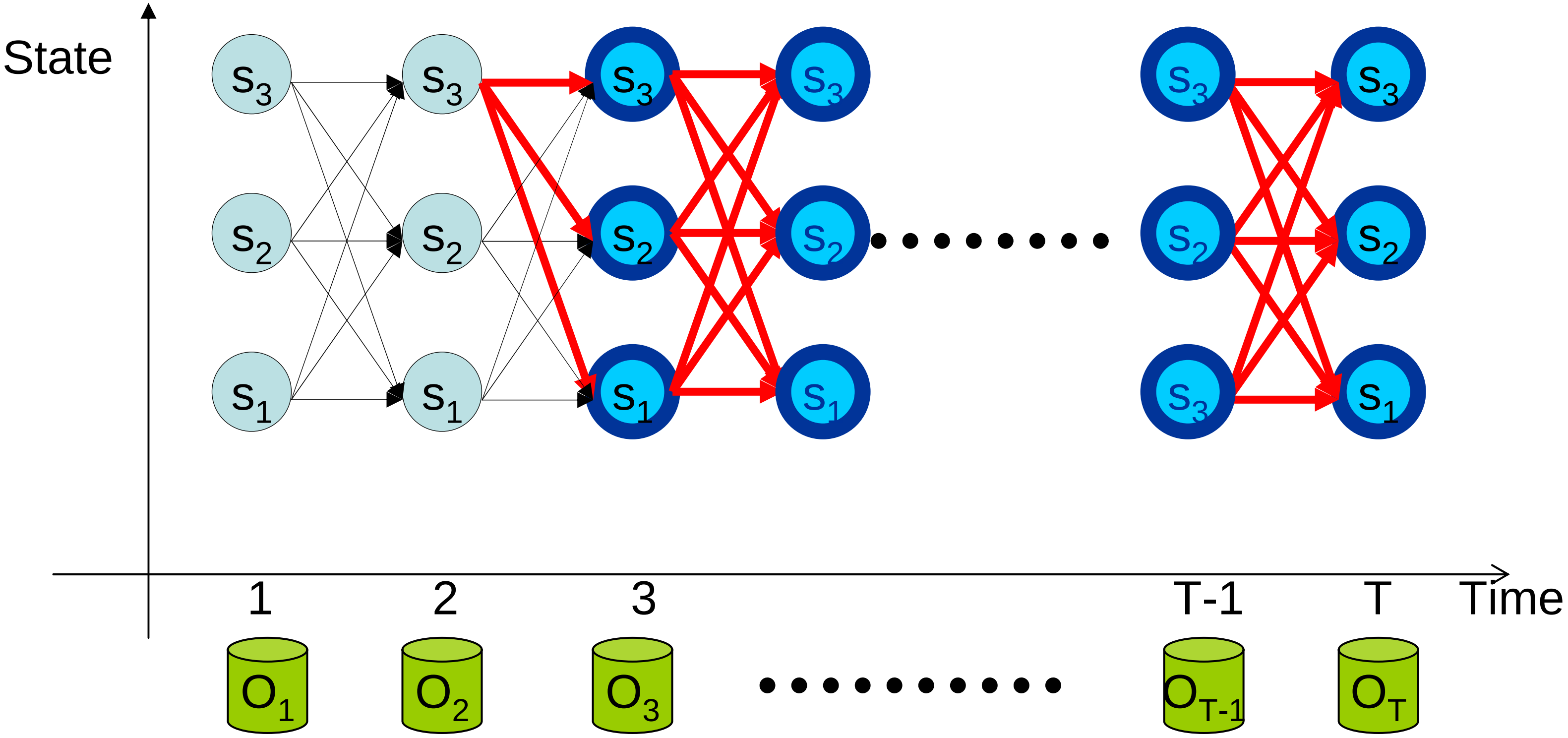
$$
\begin{aligned}
&\alpha_t(i)\,\beta_t(i) \\
&= P(\boldsymbol{o}_1, \boldsymbol{o}_2, ..., \boldsymbol{o}_t, s_t = i|\lambda) \cdot P(\boldsymbol{o}_{t+1}, \boldsymbol{o}_{t+2}, ..., \boldsymbol{o}_T|s_t = i, \lambda) \\
&= P(\boldsymbol{o}_1, \boldsymbol{o}_2, ..., \boldsymbol{o}_t|s_t = i, \lambda) P(s_t = i|\lambda) P(\boldsymbol{o}_{t+1}, \boldsymbol{o}_{t+2}, ..., \boldsymbol{o}_T|s_t = i, \lambda) \\
&= P(\boldsymbol{o}_1, ..., \boldsymbol{o}_t, ..., \boldsymbol{o}_T|s_t = i, \lambda) P(s_t = i|\lambda) \\
&= P(\boldsymbol{o}_1, ..., \boldsymbol{o}_t, ..., \boldsymbol{o}_T, s_t = i|\lambda) \\
&= P(\boldsymbol{O}, s_t = i|\lambda)
\end{aligned}
$$

- $P(\boldsymbol{O}|\lambda) = \sum_{i=1}^{N} P(\boldsymbol{O}, s_t = i|\lambda) = \sum_{i=1}^{N} \alpha_t(i)\,\beta_t(i)$

Basic Problem 1 of HMM

- The Backward Procedure

- $\beta_2(3)=P(o_3,o_4,\ldots, o_T|s_2=3,\lambda)$
  $=a_{31}* b_1(o_3)*\beta_3(1) +a_{32}* b_2(o_3)*\beta_3(2)+a_{33}* b_1(o_3)*\beta_3(3)$



Basic Problem 2 of HMM

How to choose an optimal state sequence $S=(s_1,s_2,\ldots\ldots, s_T)$?

- The first optimal criterion: **Choose the states s_t are *individually* most likely at each time *t***

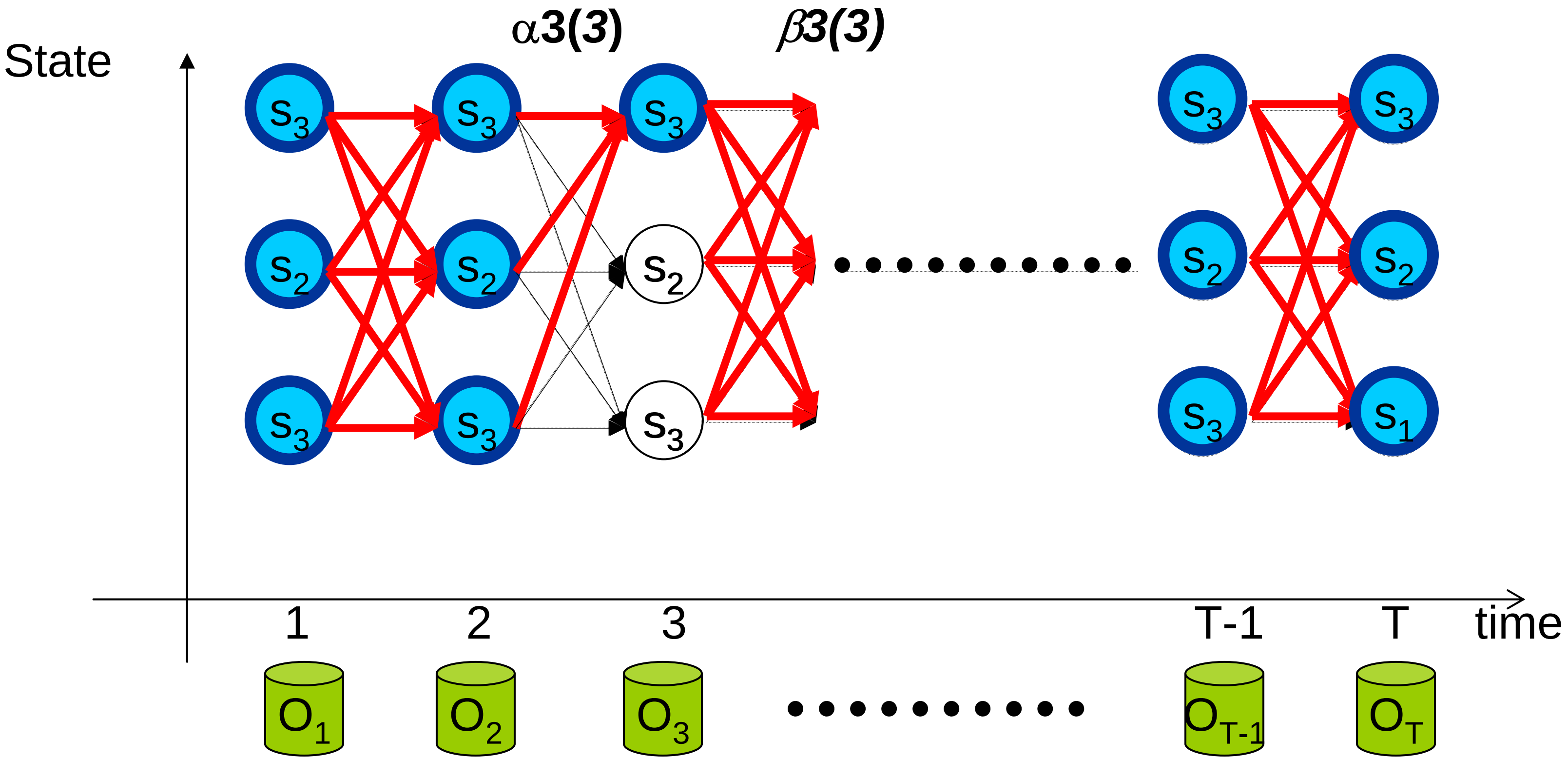
Define a posteriori probability variable $\gamma_t(i) = P(s_t = i | \boldsymbol{O}, \boldsymbol{\lambda})$

$$\gamma_t(i) = \frac{P(s_t = i, \boldsymbol{O} | \boldsymbol{\lambda})}{P(\boldsymbol{O} | \boldsymbol{\lambda})} = \frac{P(s_t = i, \boldsymbol{O} | \boldsymbol{\lambda})}{\sum_{m=1}^{N} P(s_t = m, \boldsymbol{O} | \boldsymbol{\lambda})} = \frac{\alpha_t(i)\,\beta_t(i)}{\sum_{m=1}^{N} \alpha_t(m)\,\beta_t(m)}$$

  - Solution : $s_t^* = \arg_i \max [\gamma_t(i)],\ 1 \le t \le T$
    - Problem: maximizing the probability at each time *t* individually $\boldsymbol{S}^* = s_1^* s_2^* \ldots s_T^*$ may not be a valid sequence (e.g. $a_{s_t^* s_{t+1}^*} = 0$)

Basic Problem 2 of HMM

- $P(s_3 = 3, \mathbf{O} \mid \lambda) = \alpha_3(3) * \beta_3(3)$



Basic Problem 2 of HMM

- The Viterbi Algorithm

- The second optimal criterion: The Viterbi algorithm can be regarded as the dynamic programming algorithm applied to the HMM or as a modified forward algorithm
  - Instead of summing up probabilities from different paths coming to the same destination state, the Viterbi algorithm picks and remembers the best path
    - Find a single optimal state sequence $\boldsymbol{S}=(s_1,s_2,\ldots\ldots, s_T)$
  - The Viterbi algorithm also can be illustrated in a trellis framework similar to the one for the forward algorithm

Basic Problem 2 of HMM
- The Viterbi Algorithm

- Algorithm

  Find a best state sequence $\boldsymbol{S}=(s_1,s_2,..,s_T)$ for a given observation $\boldsymbol{O}=(\boldsymbol{o}_1,\boldsymbol{o}_2,..,\boldsymbol{o}_T)$?

  Define a new variable

$$\delta_t(i)=\max_{s_1,s_2,..,s_{t-1}} P\left[s_1,s_2,..,s_{t-1},s_t=i,\boldsymbol{o}_1,\boldsymbol{o}_2,..,\boldsymbol{o}_t\middle|\lambda\right]$$

$$=\text{the best score along a single path at time } t\text{, which accounts for the first } t \text{ observation and ends in state } i$$

  By induction $\therefore \delta_{t+1}(j)=\left[\max_{1\le i\le N}\delta_t(i)a_{ij}\right]b_j(\boldsymbol{o}_{t+1})$

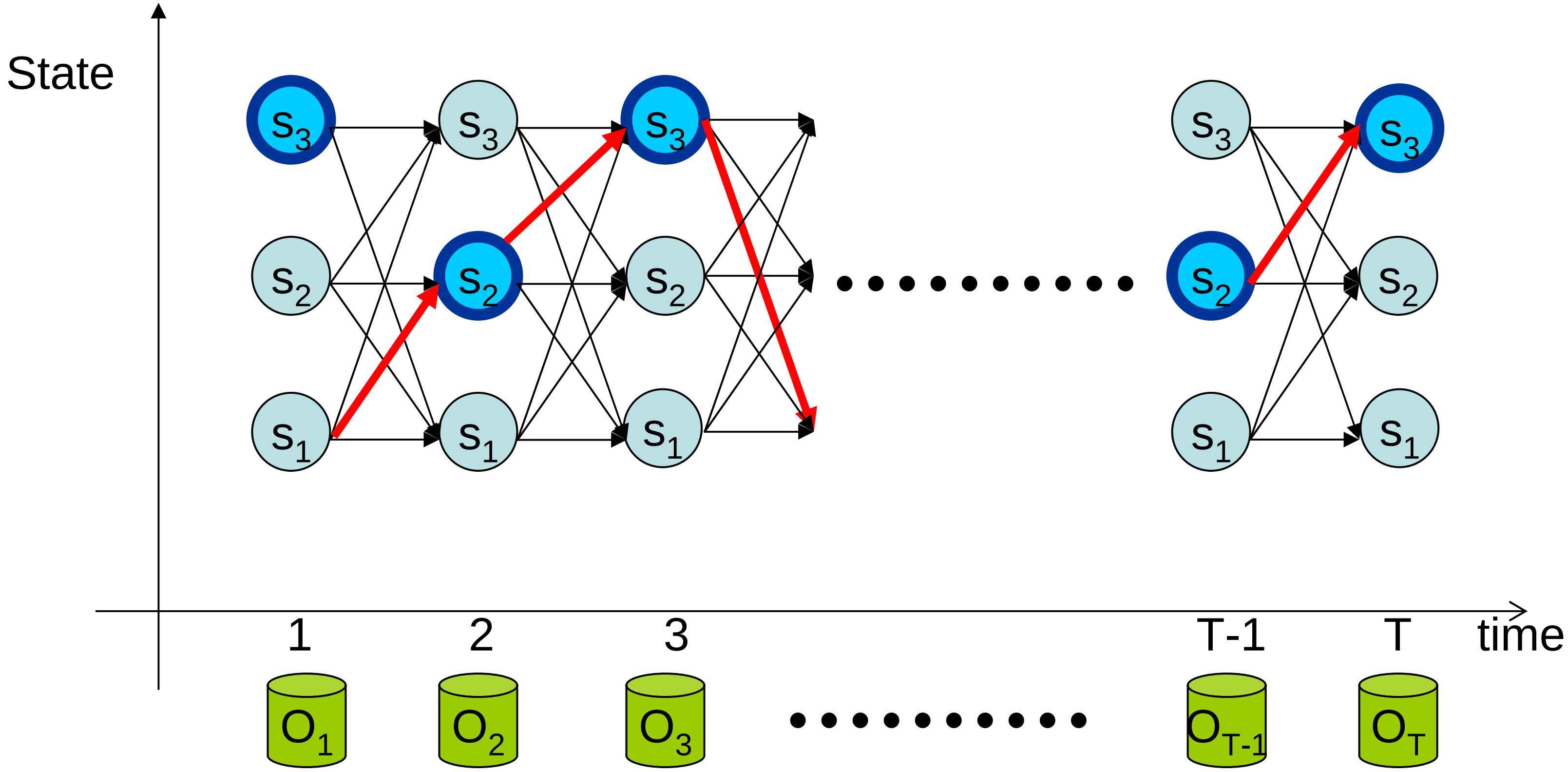
$$\psi_{t+1}(j)=\arg\max_{1\le i\le N}\delta_t(i)a_{ij} \quad \text{.... For backtracing}$$

  We can backtrace from $s_T^*=\arg\max_{1\le i\le N}\delta_T(i)$

  – Complexity: *O(N²T)*

Basic Problem 2 of HMM

- The Viterbi Algorithm



Basic Problem 2 of HMM
- The Viterbi Algorithm

- A three-state Hidden Markov Model for the *Dow Jones Industrial average*

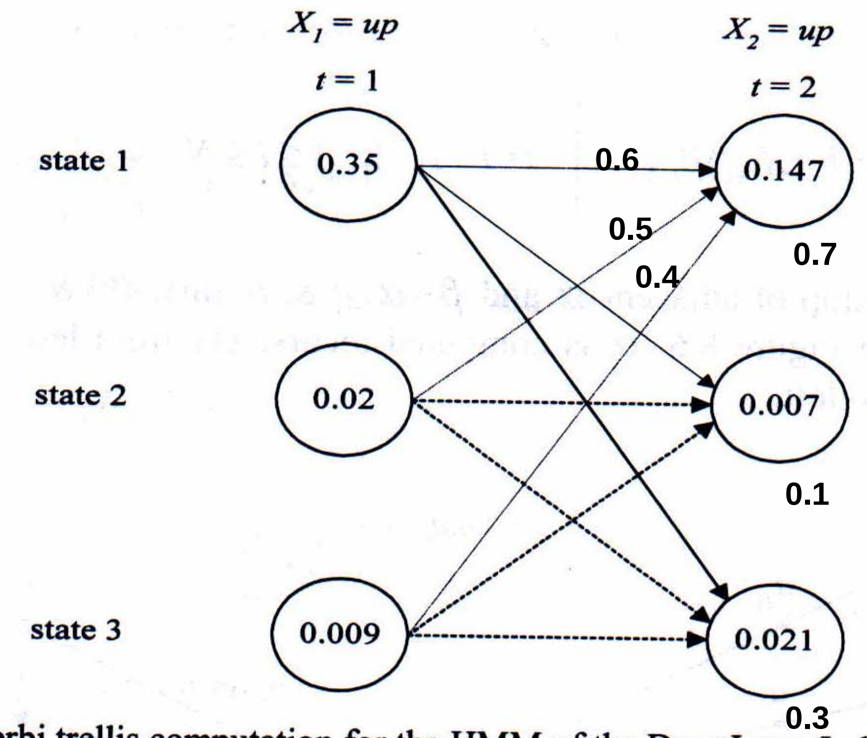


Figure 8.5 The Viterbi trellis computation for the HMM of the Dow Jones Industrial average.

Basic Problem 2 of HMM
- The Viterbi Algorithm

- Algorithm in the logarithmic form

Find a best state sequence $\boldsymbol{S}=\left(s_1,s_2,..,s_T\right)$ for a given observation $\boldsymbol{O}=\left(\boldsymbol{o}_1,\boldsymbol{o}_2,..,\boldsymbol{o}_T\right)$?

Define a new variable

$$\delta_t(i)=\max_{s_1,s_2,..,s_{t-1}} \log P\left[s_1,s_2,..,s_{t-1},s_t=i,\boldsymbol{o}_1,\boldsymbol{o}_2,..,\boldsymbol{o}_t \middle| \lambda\right]$$

= the best score along a single path at time t, which accounts for the first t observation and ends in state i

By induction $\therefore \delta_{t+1}(j)=\left[\max_{1\le i\le N}\left(\delta_t(i)+\log a_{ij}\right)\right]+\log b_j\left(\boldsymbol{o}_{t+1}\right)$

$$\psi_{t+1}(j)=\arg\max_{1\le i\le N}\left(\delta_t(i)+\log a_{ij}\right) \text{ .... For backtracing}$$

We can backtrace from $s_T^*=\arg\max_{1\le i\le N}\delta_T(i)$

Homework-2

- A three-state Hidden Markov Model for the *Dow Jones Industrial average*

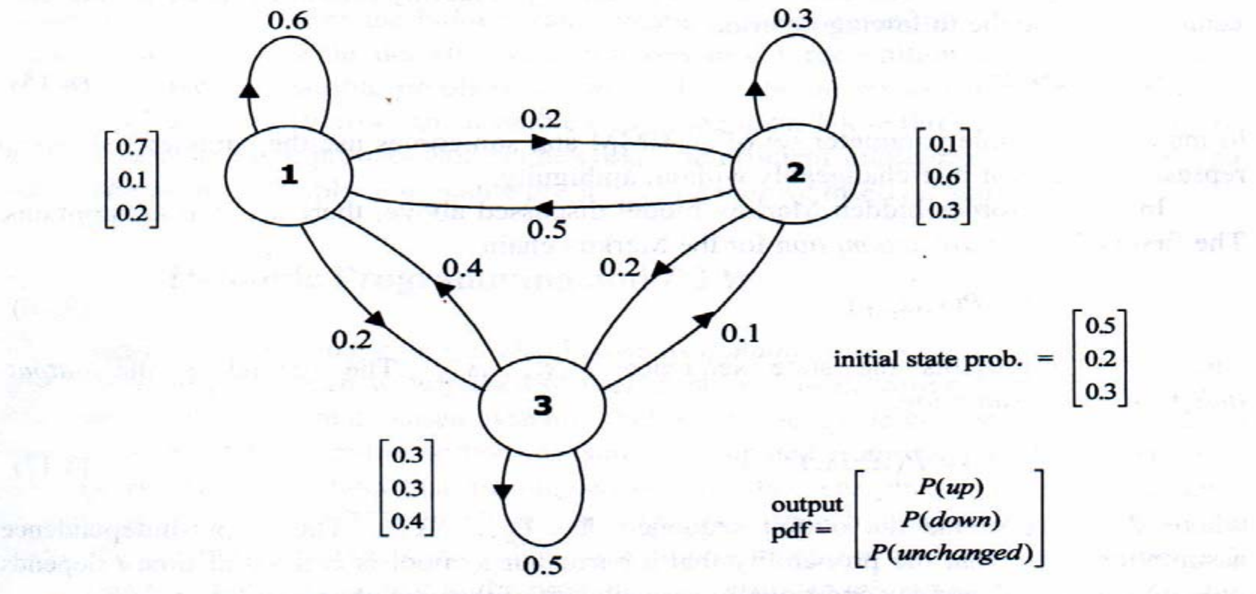


Figure 8.2 A hidden Markov model for the Dow Jones Industrial average. The three states no longer have deterministic meanings as in the Markov chain illustrated in Figure 8.1.

- *Find the probability:*

  $P(up, up, unchanged, down, unchanged, down, up|\lambda)$

- *Fnd the optimal state sequence of the model which generates the observation sequence: (up, up, unchanged, down, unchanged, down, up)*

Probability Addition in F-B Algorithm

- In Forward-backward algorithm, operations usually implemented in logarithmic domain

- Assume that we want to add P_1 and P_2

$$\text{if } P_1 \geq P_2$$

$$\log_b(P_1 + P_2) = \log P_1 + \log_b\left(1 + b^{\log_b P_2 - \log_b P_1}\right)$$

$$\text{else}$$

$$\log_b(P_1 + P_2) = \log P_2 + \log_b\left(1 + b^{\log_b P_1 - \log_b P_2}\right)$$

The values of $\log_b(1 + b^x)$ can be saved in in a table to speedup the operations

Probability Addition in F-B Algorithm

- An example code

```
#define LZERO  (-1.0E10)   // ~log(0)
#define LSMALL (-0.5E10)   // log values < LSMALL are set to LZERO
#define minLogExp  -log(-LZERO)

double LogAdd(double x, double y)
{
 double temp,diff,z;
   if (x<y)
   {
      temp = x; x = y; y = temp;
   }
   diff = y-x;
   if (diff<minLogExp)
      return  (x<LSMALL) ?  LZERO:x;
   else
    {
      z = exp(diff);
      return x+log(1.0+z);
   }
}
```

Basic Problem 3 of HMM

Intuitive View

- How to adjust (re-estimate) the model parameter $\lambda=(\boldsymbol{A},\boldsymbol{B},\pi)$ to maximize $P(\boldsymbol{O}|\lambda)$?
  - The most difficult of the three problems, because there is no known analytical method that maximizes the joint probability of the training data in a close form
  - The data is incomplete because of the hidden state sequences
  - Well-solved by the *Baum-Welch* (known as *forward-backward*) algorithm and *EM* (*Expectation Maximization*) algorithm
    - Iterative update and improvement

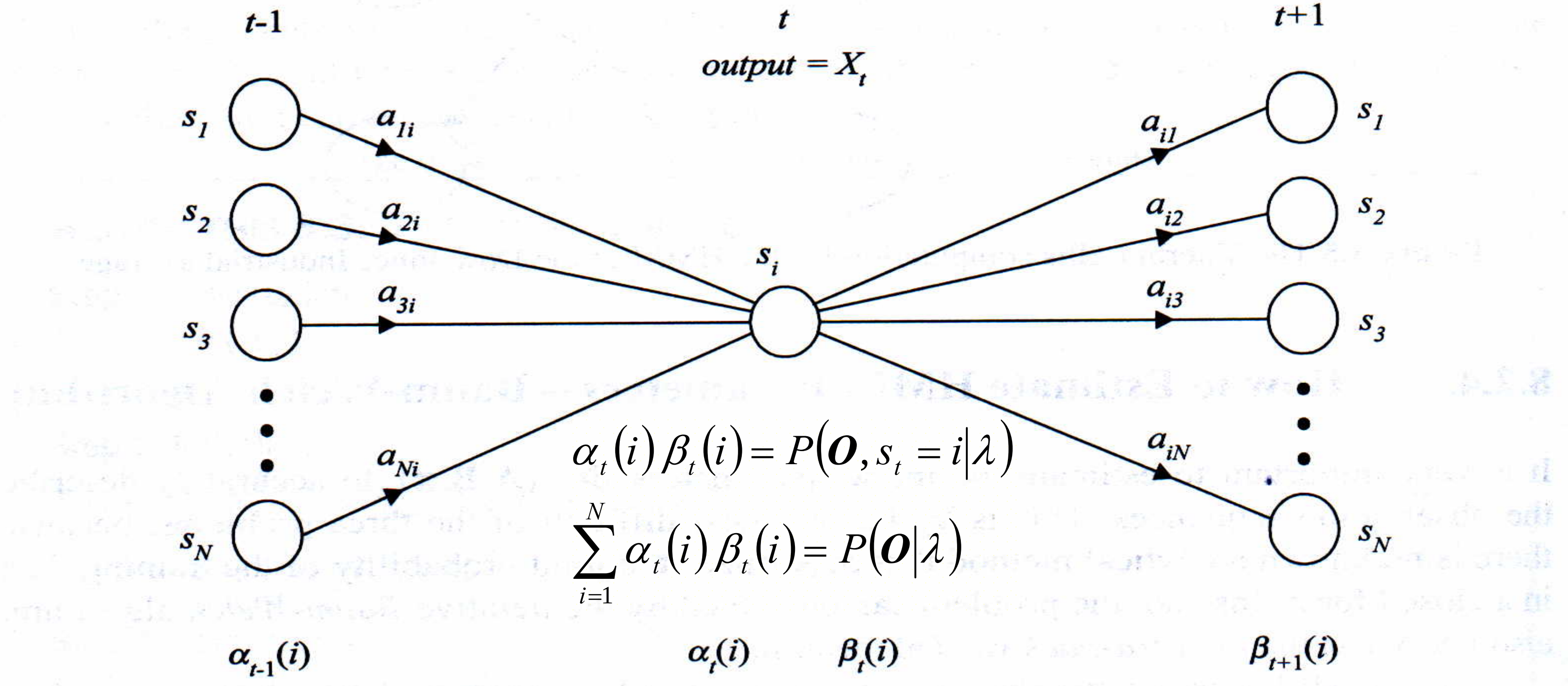
Basic Problem 3 of HMM

Intuitive View

- Relation between the forward and backward variables

$$\alpha_t(i) = P(\boldsymbol{o}_1, \boldsymbol{o}_2, \ldots, \boldsymbol{o}_t, s_t = i | \lambda)$$
$$= \left[\sum_{j=1}^{N} \alpha_{t-1}(j) a_{ji} \right] b_i(\boldsymbol{o}_t)$$

$$\beta_t(i) = P(\boldsymbol{o}_{t+1}, \boldsymbol{o}_{t+2}, \ldots, \boldsymbol{o}_T | s_t = i, \lambda)$$
$$= \sum_{j=1}^{N} \beta_{t+1}(j) b_j(\boldsymbol{o}_{t+1}) a_{ij}$$



$$\alpha_t(i)\beta_t(i) = P(\boldsymbol{O}, s_t = i | \lambda)$$

$$\sum_{i=1}^{N} \alpha_t(i)\beta_t(i) = P(\boldsymbol{O} | \lambda)$$

Figure 8.6 The relationship of α_{t-1} and α_t and β_t and β_{t+1} in the forward-backward algorithm.

Basic Problem 3 of HMM

Intuitive View

- Define a new variable:

$$\xi_t(i,j) = P\left(s_t = i, s_{t+1} = j \middle| \boldsymbol{O}, \boldsymbol{\lambda}\right)$$

  - Probability being at state *i* at time *t* and at state *j* at time *t+1*

$$\xi_t(i,j) = \frac{P\left(s_t = i, s_{t+1} = j, \boldsymbol{O} \middle| \boldsymbol{\lambda}\right)}{P\left(\boldsymbol{O} \middle| \boldsymbol{\lambda}\right)}$$

$$= \frac{\alpha_t(i) a_{ij} b_j\left(\boldsymbol{o}_{t+1}\right) \beta_{t+1}(j)}{P\left(\boldsymbol{O} \middle| \boldsymbol{\lambda}\right)} = \frac{\alpha_t(i) a_{ij} b_j\left(\boldsymbol{o}_{t+1}\right) \beta_{t+1}(j)}{\sum_{m=1}^{N} \sum_{n=1}^{N} \alpha_t(m) a_{mn} b_n\left(\boldsymbol{o}_{t+1}\right) \beta_{t+1}(n)}$$

- Recall the posteriori probability variable:

$$\gamma_t(i) = \sum_{j=1}^{N} P\left(s_t = j \middle| \boldsymbol{O}, \lambda\right)$$

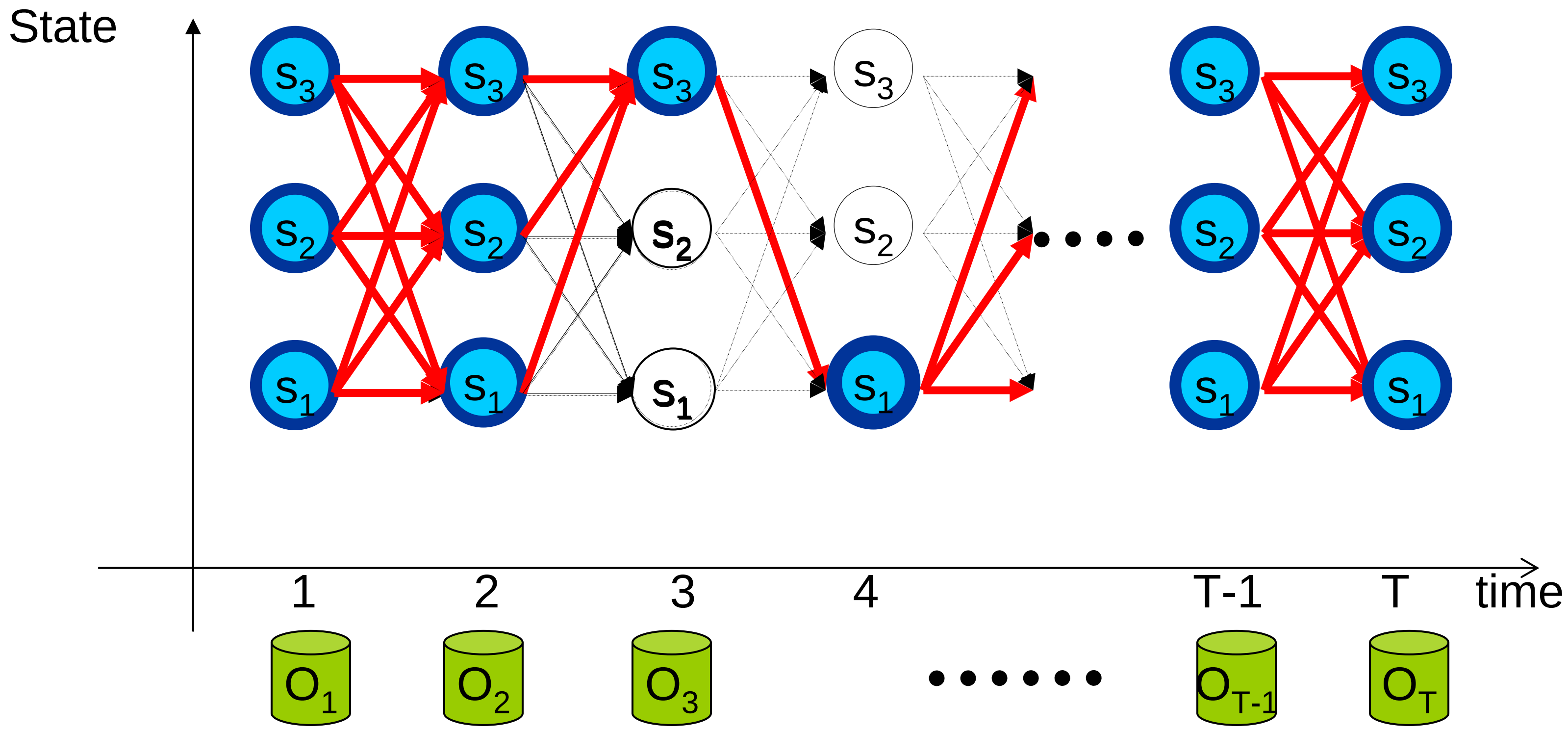
Note: $\gamma_t(i)$ also can be represented as $\frac{\alpha_t(i)\beta_t(i)}{\sum_{m=1}^{N} \alpha_t(m)\beta_t(m)}$

$$\gamma_t(i) = \sum_{j=1}^{N} P\left(s_t = i, s_{t+1} = j \middle| \boldsymbol{O}, \lambda\right) = \sum_{j=1}^{N} \xi_t(i,j)$$

Basic Problem 3 of HMM

Intuitive View

- $P(s_3 = 3, s_4 = 1, \mathbf{O} \mid \lambda) = \alpha_3(3) * a_{31} * b_1(o_4) * \beta_1(4)$



Basic Problem 3 of HMM

Intuitive View

- $\xi_t(i,j) = P(s_t = i, s_{t+1} = j | \boldsymbol{O}, \boldsymbol{\lambda})$

  $$\sum_{t=1}^{T-1} \xi_t(i,j) = \text{expected number of transitions from state } i \text{ to state } j \text{ in } \boldsymbol{O}$$

- $\gamma_t(i) = \sum_{j=1}^{N} P(s_t = j | \boldsymbol{O}, \lambda)$

  $$\sum_{t=1}^{T-1} \gamma_t(i) = \sum_{t=1}^{T-1} \sum_{j=1}^{N} \xi_t(i,j) = \text{expected number of transitions from state } i \text{ in } \boldsymbol{O}$$

- A set of reasonable re-estimation formula for {***A***, π} is

$$\bar{\pi}_i = \text{expected freqency (number of times) in state } i \text{ at time } t = 1$$
$$= \gamma_1(i)$$

$$\bar{a}_{ij} = \frac{\text{expected number of transition from state } i \text{ to state } j}{\text{expected number of transition from state } i} = \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{t=1}^{T-1} \gamma_t(i)}$$

Basic Problem 3 of HMM

Intuitive View

- A set of reasonable re-estimation formula for {*B*} is
  - For Discrete and finite observation $b_j(\mathbf{v}_k)=P(\mathbf{o}_t=\mathbf{v}_k|\mathbf{s}_t=j)$

$$\bar{b}_j(\boldsymbol{v}_k)=\bar{P}(\boldsymbol{o}=\boldsymbol{v}_k|s=j)=\frac{\text{expected number of times in state } j \text{ and observing symbol } \boldsymbol{v}_k}{\text{expected number of times in state } j}=\frac{\sum_{\substack{\mathrm{t}=1 \\ \text{such that } \boldsymbol{o}=\boldsymbol{v}_k}}^{\mathrm{T}}\gamma_t(j)}{\sum_{\mathrm{t}=1}^{\mathrm{T}}\gamma_t(j)}$$

  - For continuous and infinite observation $b_j(\mathbf{v})=f_{\mathbf{O}|\mathbf{S}}(\mathbf{o}_t=\mathbf{v}|\mathbf{s}_t=j)$,

$$\bar{b}_j(\boldsymbol{v})=\sum_{k=1}^{M}\bar{c}_{jk}N(\boldsymbol{v};\bar{\boldsymbol{\mu}}_{jk},\bar{\boldsymbol{\Sigma}}_{jk})=\sum_{k=1}^{M}\bar{c}_{jk}\left(\frac{1}{\left(\sqrt{2\pi}\right)^{L}\left|\bar{\boldsymbol{\Sigma}}_{jk}\right|^{1/2}}\exp\left(-\frac{1}{2}\left(\boldsymbol{v}-\bar{\boldsymbol{\mu}}_{jk}\right)^{t}\bar{\Sigma}_{jk}^{-1}\left(\boldsymbol{v}-\bar{\boldsymbol{\mu}}_{jk}\right)\right)\right)$$

Modeled as a mixture of multivariate Gaussian distributions

Basic Problem 3 of HMM

Intuitive View

- For continuous and infinite observation (Cont.)
  - Define a new variable $\gamma_t(j,k)$
    - $\gamma_t(j,k)$ is the probability of being in state *j* at time *t* with the *k*-th mixture component accounting for $\boldsymbol{o}_t$

$$\gamma_t(j,k) = P(s_t = j, m_t = k \mid \boldsymbol{O}, \boldsymbol{\lambda})$$

$$= P(s_t = j \mid \boldsymbol{O}, \boldsymbol{\lambda}) P(m_t = k \mid s_t = j, \boldsymbol{O}, \boldsymbol{\lambda})$$

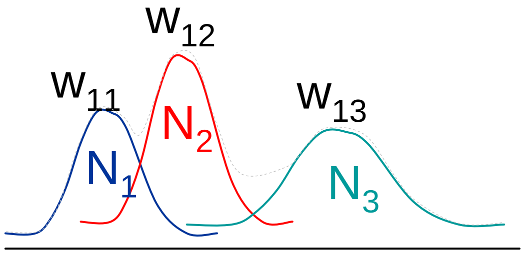
$$= \gamma_t(j) P(m_t = k \mid s_t = j, \boldsymbol{O}, \boldsymbol{\lambda})$$

$$= \gamma_t(j) \frac{P(m_t = k \mid s_t = j, \boldsymbol{\lambda}) P(\boldsymbol{O} \mid s_t = j, m_t = k, \boldsymbol{\lambda})}{P(\boldsymbol{O} \mid s_t = j, \boldsymbol{\lambda})}$$

(observation-independent assumption is applied)

……

$$= \left[\frac{\alpha_t(j)\beta_t(j)}{\sum_{s=1}^{N} \alpha_t(s)\beta_t(s)} \right] \left[\frac{c_{jk} N(\boldsymbol{o}_t; \boldsymbol{\mu}_{jk}, \boldsymbol{\Sigma}_{jk})}{\sum_{m=1}^{M} c_{jm} N(\boldsymbol{o}_t; \boldsymbol{\mu}_{jm}, \boldsymbol{\Sigma}_{jm})} \right]$$



Distribution for State 1

$$\text{Note}: \gamma_t(j) = \sum_{m=1}^{M} \gamma_t(j,m)$$

Basic Problem 3 of HMM

Intuitive View

– For Continuous and infinite observation (Cont.)

$$\overline{c}_{jk} = \frac{\text{expected number of times in state } j \text{ and mixture } k}{\text{expected number of times in state } j} = \frac{\sum_{t=1}^{T} \gamma_t(j,k)}{\sum_{t=1}^{T}\sum_{m=1}^{M} \gamma_t(j,m)}$$

$$\overline{\boldsymbol{\mu}}_{jk} = \text{weighted average (mean) of observations at state } j \text{ and mixture } k = \frac{\sum_{t=1}^{T} \gamma_t(j,k)\cdot \boldsymbol{o}_t}{\sum_{t=1}^{T} \gamma_t(j,k)}$$

$$\overline{\boldsymbol{\Sigma}}_{jk} = \text{weighted covariance of observations at state } j \text{ and mixture } k$$

$$= \frac{\sum_{t=1}^{T} \gamma_t(j,k)\cdot \left(\boldsymbol{o}_t - \overline{\boldsymbol{\mu}}_{jk}\right)\left(\boldsymbol{o}_t - \overline{\boldsymbol{\mu}}_{jk}\right)^t}{\sum_{t=1}^{T} \gamma_t(j,k)}$$

Semicontinuous HMMs

- The HMM state mixture density functions are tied together across all the models to form a set of shared kernels
  - The semicontinuous or tied-mixture HMM

$$b_j(\boldsymbol{o}) = \sum_{k=1}^{M} b_j(k) f(\boldsymbol{o}|v_k) = \sum_{k=1}^{M} \underline{b_j(k)} \underline{N(\boldsymbol{o}, \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)}$$

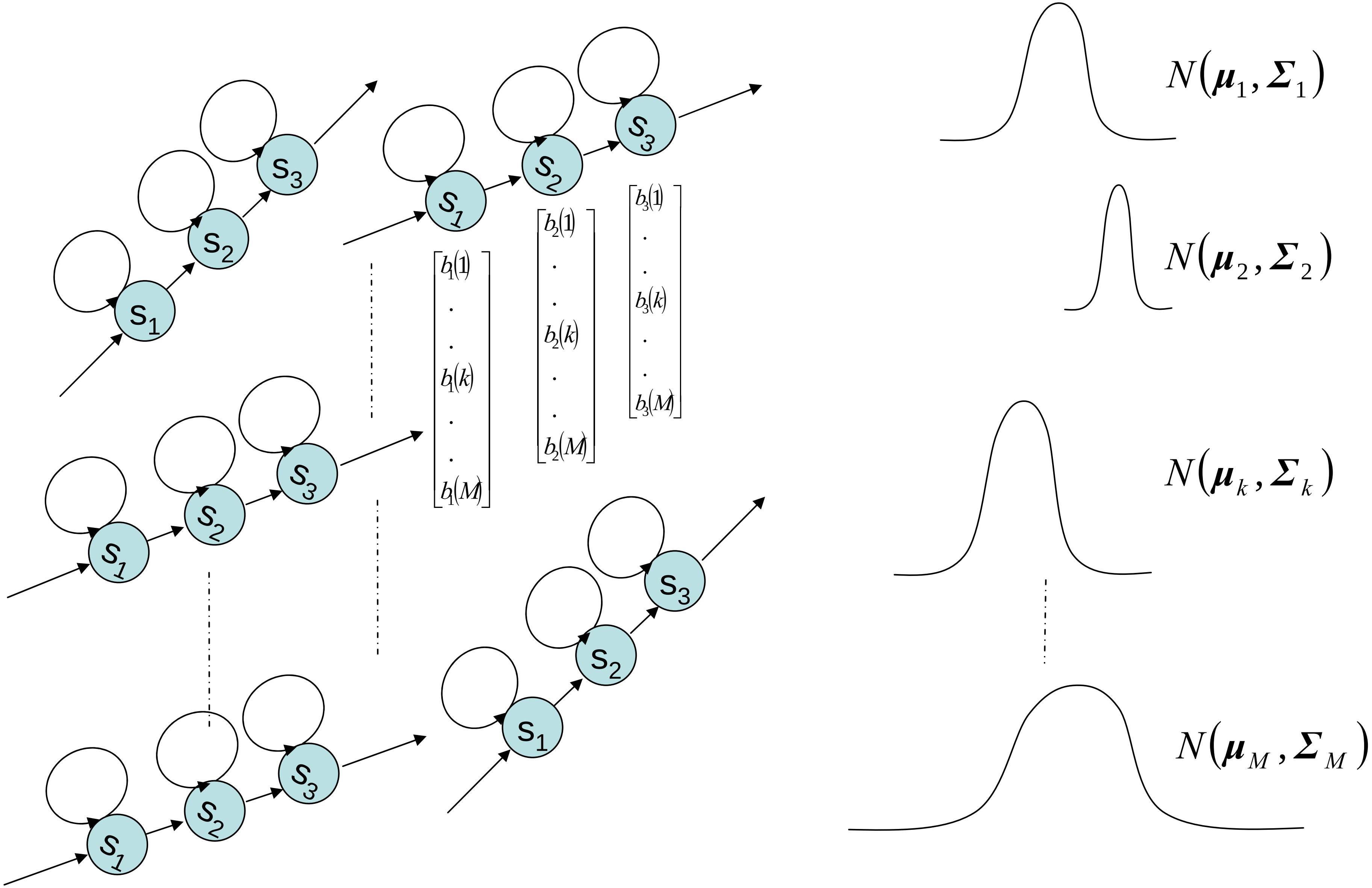
state output
Probability of state *j*

k-th mixture weight
t of state *j*
(discrete, model-dependent)

k-th mixture density function or *k*-th codeword
(shared across HMMs, *M* is very large)

  - A combination of the discrete HMM and the continuous HMM
    - A combination of *discrete* model-dependent weight coefficient and *continuous* codebook probability density function
  - Because *M* is large, we can simply use the *L* most significant values $f(\boldsymbol{o}|v_k)$
    - Experience showed that *L* is *1~3%* of *M* is adequate
  - Partial tying of $f(\boldsymbol{o}|v_k)$ for different phonetic class

Semicontinuous HMMs



Initialization of HMM

- A good initialization of HMM training :
  Segmental K-Means Segmentation into States
  - Assume that we have a training set of observations and an initial estimate of all model parameters
  - Step 1 : The set of training observation sequences is segmented into states, based on the initial model (finding the optimal state sequence by *Viterbi* Algorithm)
  - Step 2 : For discrete density HMM (using M-codeword codebook)

$$\bar{b}_j(k) = \frac{\text{the number of vectors with codebook index } k \text{ in state } j}{\text{the number of vectors in state } j}$$

  - For continuous density HMM (M Gaussian mixtures per state)

$$\Rightarrow \text{cluster the observation vectors within each state } j \text{ into a set of } M \text{ clusters}$$

$$\bar{w}_{jm} = \text{number of vectors classified in cluster } m \text{ of state } j$$

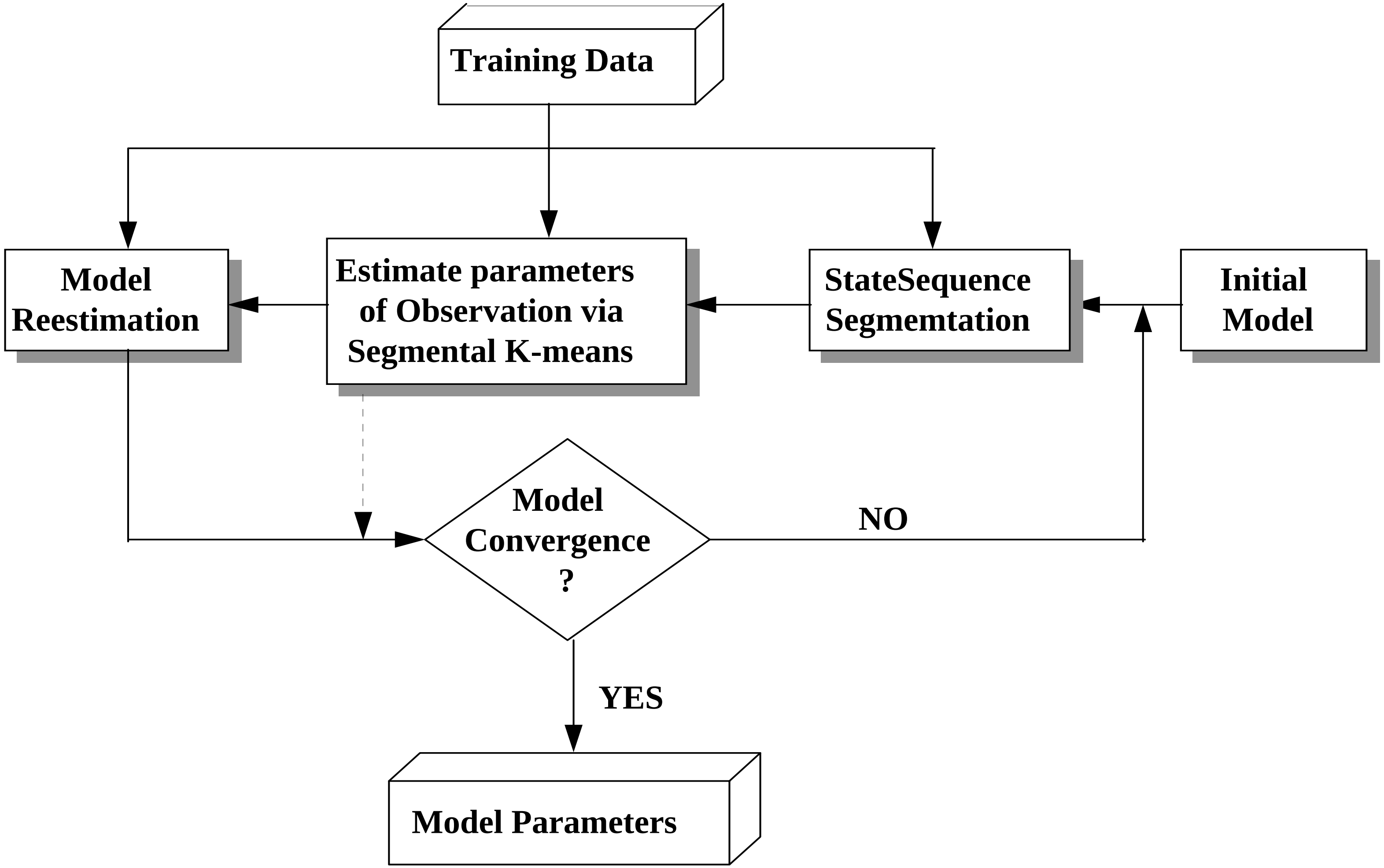
$$\text{divided by the number of vectors in state } j$$

$$\bar{\mu}_{jm} = \text{sample mean of the vectors classified in cluster } m \text{ of state } j$$

$$\bar{\Sigma}_{jm} = \text{sample covariance matrix of the vectors classified in cluster } m \text{ of state } j$$

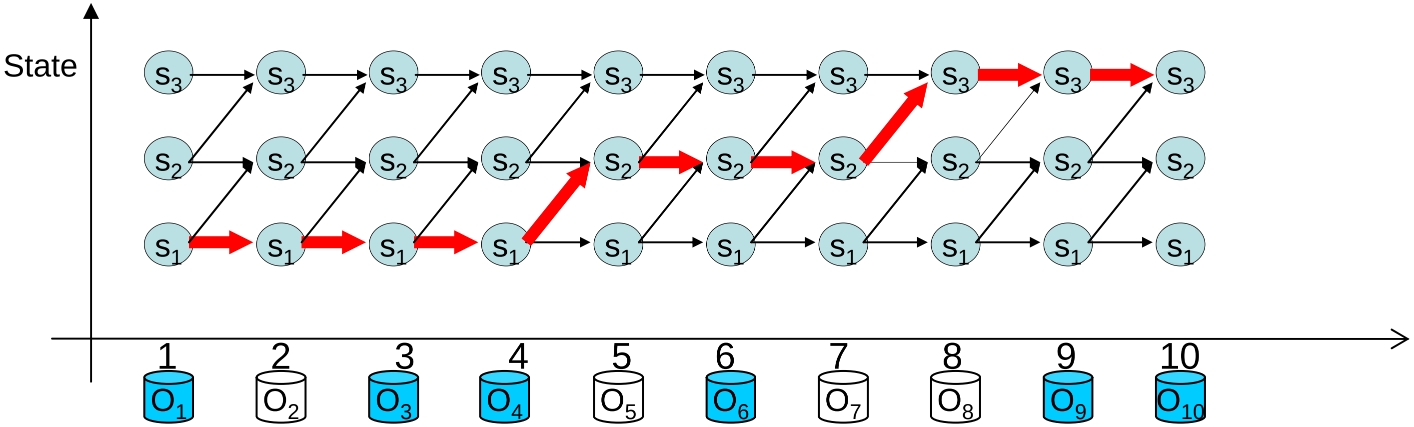
  - Step 3: Evaluate the model score.
    If the difference between the previous and current model scores is greater than a threshold, go back to Step 1, otherwise stop the initial model is generated

Initialization of HMM

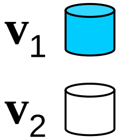


Initialization of HMM

- An example for discrete HMM
  - 3 states and 2 codeword

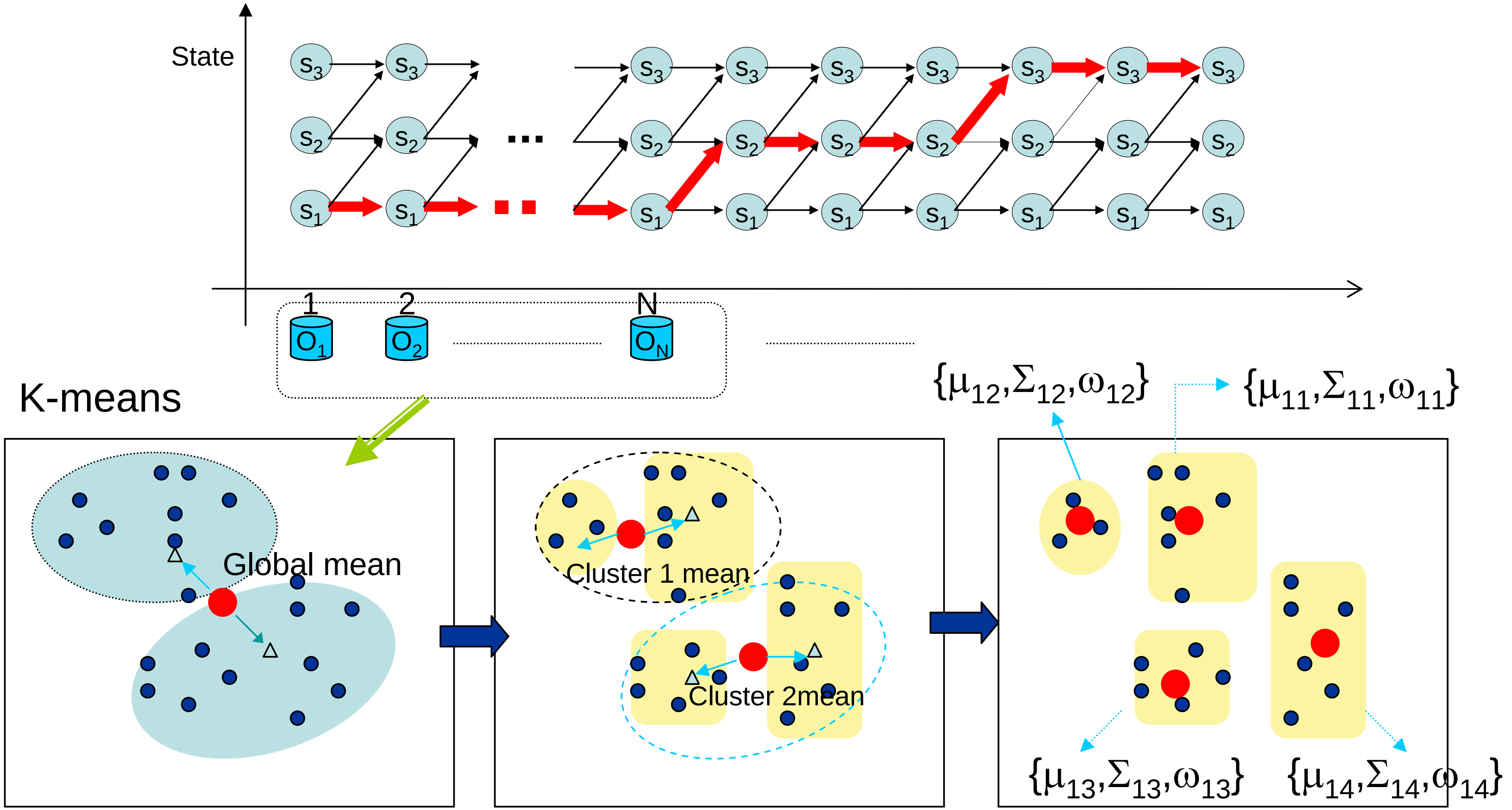


- $b_1(\mathbf{v}_1)=3/4$, $b_1(\mathbf{v}_2)=1/4$
- $b_2(\mathbf{v}_1)=1/3$, $b_2(\mathbf{v}_2)=2/3$
- $b_3(\mathbf{v}_1)=2/3$, $b_3(\mathbf{v}_2)=1/3$



Initialization of HMM

- An example for Continuous HMM
    - 3 states and 4 Gaussian mixtures per state



HMM Topology

- Speech is time-evolving non-stationary signal
  - Each HMM state has the ability to capture some quai-stationary segment in the non-stationary speech signal
  - A *left-to-right* topology is a natural candidate to model the speech signal

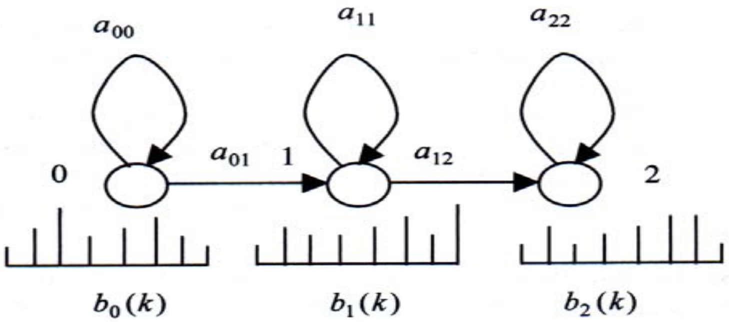


Figure 8.8 A typical hidden Markov model used to model phonemes. There are three states (0-2) and each state has an associated output probability distribution.

  - It is general to represent a phone using 3~5 states (English) and a syllable using 6~8 states (Mandarin Chinese)

HMM Limitations

- The assumptions of conventional HMMs in Speech Processing
  - The state duration follows an exponential distribution
    - Don’t provide adequate representation of the temporal structure of speech

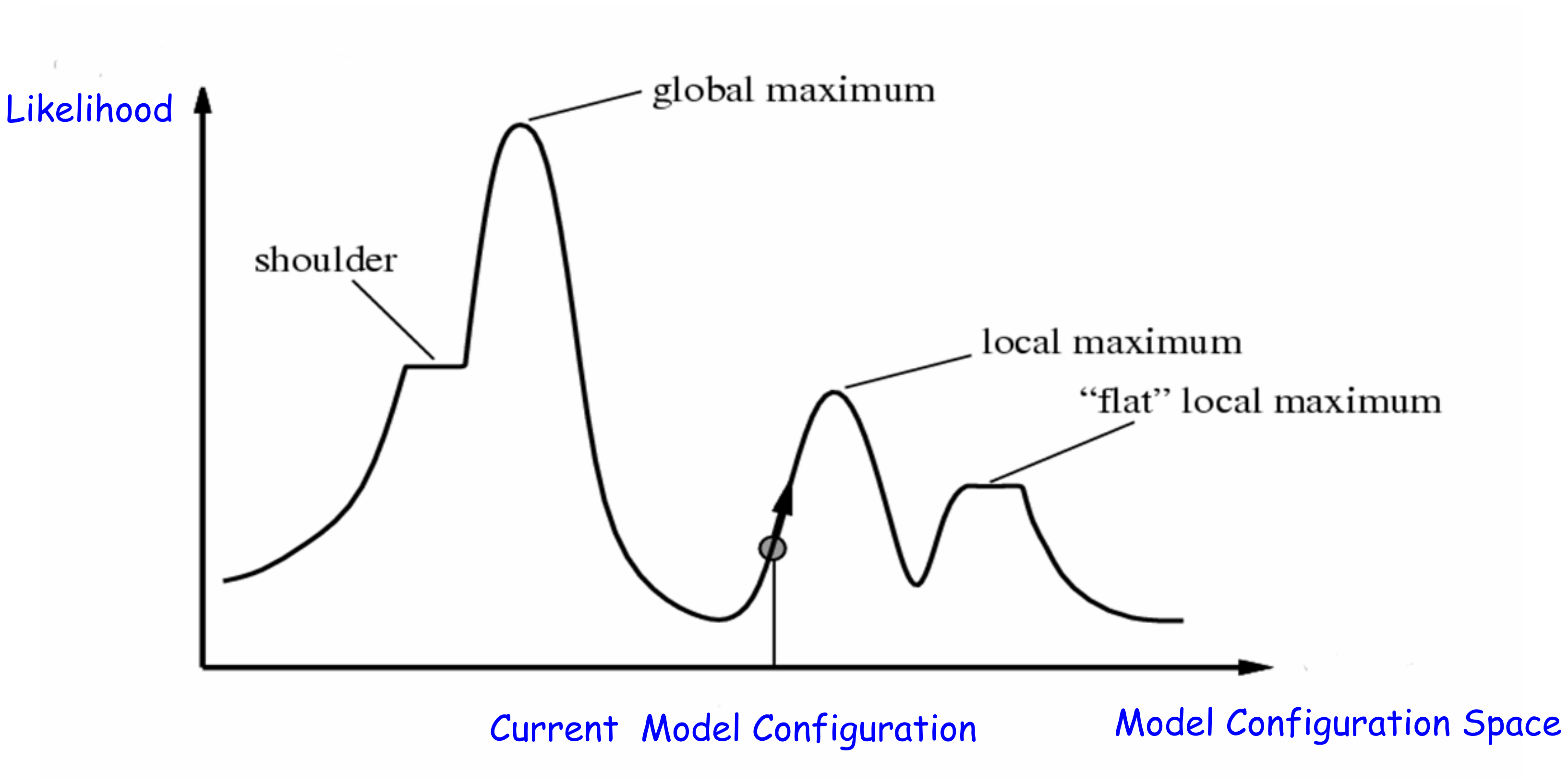
$$d_i(t) = a_{ii}^{t}(1 - a_{ii})$$

  - **First order (Markov) assumption:** the state transition depends only on the origin and destination
  - **Output-independent assumption:** all observation frames are dependent on the state that generated them, not on neighboring observation frames

Researchers have proposed a number of techniques to address these limitations, albeit these solution have not significantly improved speech recognition accuracy for practical applications.

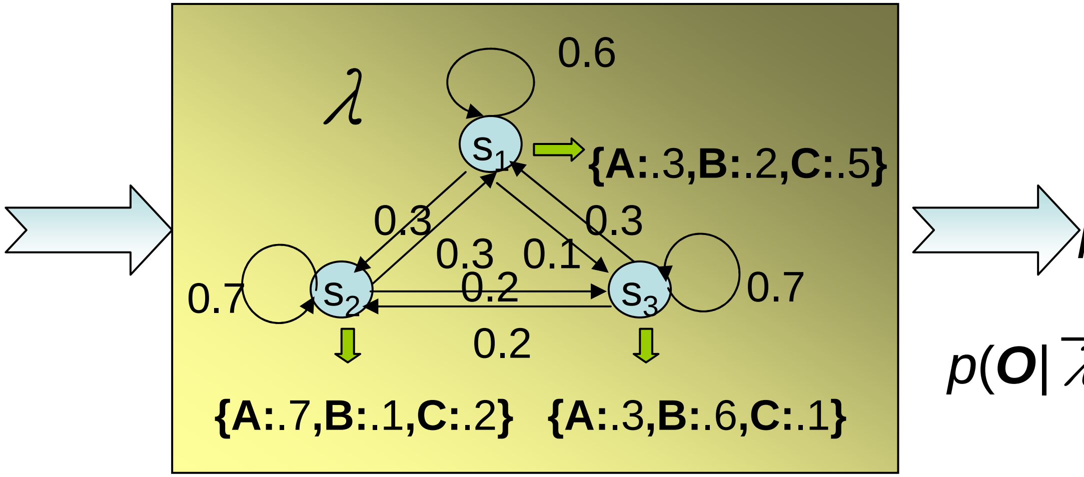
HMM Limitations

- The HMM parameters trained by the *Baum-Welch* algorithm and *EM* algorithm were only locally optimized



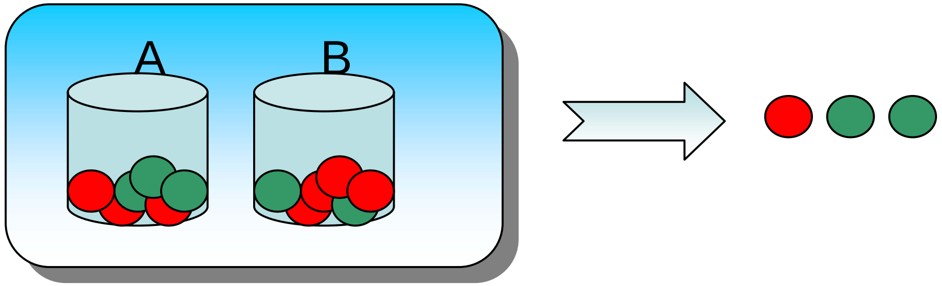
The EM Algorithm

$o_1, o_2, \ldots\ldots, o_T$



$p(\boldsymbol{O}|\lambda)$

$p(\boldsymbol{O}|\bar{\lambda}) > p(\boldsymbol{O}|\lambda)$



Observed data : ***O*** : “ball sequence”
Latent data : ***S*** : “bottle sequence”

Parameters to be estimated to maximize log$P(\boldsymbol{O}|\lambda)$
$\lambda=\{P(A),P(B),P(B|A),P(A|B),P(R|A),P(G|A),P(R|B),P(G|B)\}$

The EM Algorithm

- Introduction of EM (Expectation Maximization):
  - Why EM?
    - Simple optimization algorithms for likelihood function relies on the intermediate variables, called latent (隱藏的)data
      In our case here, ***the state sequence** is the latent data*
    - Direct access to the data necessary to estimate the parameters is impossible or difficult
      In our case here, it is almost impossible to estimate {***A***,***B***, π} without consideration of the ***state sequence***
  - Two Major Steps :
    - ***E*** : expectation with respect to the latent data using the current estimate of the parameters and conditioned on the observations $E[\bullet]_{S|\lambda,\mathbf{O}}$
    - ***M***: provides a new estimation of the parameters according to Maximum likelihood (ML) or Maximum A Posterior (MAP) Criteria

The EM Algorithm

ML and MAP

- Estimation principle based on observations:

$$\boldsymbol{x}=\left(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_n\right) \Longleftrightarrow \boldsymbol{X}=\left\{X_1,X_2,\ldots,X_n\right\}$$

  - **The Maximum Likelihood (ML) Principle**
    find the model parameter $\boldsymbol{\Phi}$ so that the likelihood $p\left(\boldsymbol{x}\middle|\boldsymbol{\Phi}\right)$ is maximum
    for example, if $\boldsymbol{\Phi}=\{\boldsymbol{\mu},\boldsymbol{\Sigma}\}$ *is the parameters of a multivariate normal distribution, and* ***X*** *is i.i.d.* (independent, identically distributed), *then the ML estimate of* $\boldsymbol{\Phi}=\{\boldsymbol{\mu},\boldsymbol{\Sigma}\}$ *is*

$$\boldsymbol{\mu}_{ML}=\frac{1}{n}\sum_{i=1}^{n}\boldsymbol{x}_i \quad , \quad \boldsymbol{\Sigma}_{ML}=\frac{1}{n}\sum_{i=1}^{n}\left(\boldsymbol{x}_i-\boldsymbol{\mu}_{ML}\right)\left(\boldsymbol{x}_i-\boldsymbol{\mu}_{ML}\right)^t$$

  - **The Maximum A Posteriori (MAP) Principle**
    find the model parameter $\boldsymbol{\Phi}$ so that the likelihood $p\left(\boldsymbol{\Phi}\middle|\boldsymbol{x}\right)$ is maximum

The EM Algorithm

- The EM Algorithm is important to HMMs and other learning techniques
  - Discover new model parameters to maximize the log-likelihood of incomplete data $\log P(\boldsymbol{O}|\boldsymbol{\lambda})$ by iteratively maximizing the expectation of log-likelihood from complete data $\log P(\boldsymbol{O},\boldsymbol{S}|\lambda)$

- Using scalar random variables to introduce the EM algorithm
  - The observable training data $\boldsymbol{O}$
    - We want to maximize $P(\boldsymbol{O}|\lambda)$, $\boldsymbol{\lambda}$ is a parameter vector
  - The hidden (unobservable) data $\boldsymbol{S}$
    - E.g. the component densities of observable data $\boldsymbol{O}$, or the underlying state sequence in HMMs

The EM Algorithm

- Assume we have λ and estimate the probability that each $\mathbf{S}$ occurred in the generation of $\mathbf{O}$
- Pretend we had in fact observed a complete data pair $(\mathbf{O},\mathbf{S})$ with frequency proportional to the probability $P(\mathbf{O},\mathbf{S}|\lambda)$, to computed a new $\bar{\lambda}$, the maximum likelihood estimate of λ
- Does the process converge?
- **Algorithm** unknown model setting

$$P(\mathbf{O},\mathbf{S}|\bar{\lambda})=P(\mathbf{S}|\mathbf{O},\bar{\lambda})P(\mathbf{O}|\bar{\lambda})$$

Bayes' rule

complete data likelihood

incomplete data likelihood

  - **Log-likelihood expression** and expectation taken over ***S***

$$\log P(\mathbf{O}|\bar{\lambda})=\log P(\mathbf{O},\mathbf{S}|\bar{\lambda})-\log P(\mathbf{S}|\mathbf{O},\bar{\lambda})$$

take expectation over **S**

$$\log P(\mathbf{O}|\bar{\lambda})=\sum_{S}\left[P(\mathbf{S}|\mathbf{O},\lambda)\log P(\mathbf{O}|\bar{\lambda})\right]$$

$$=\sum_{\mathbf{S}}\left[P(\mathbf{S}|\mathbf{O},\lambda)\log P(\mathbf{O},\mathbf{S}|\bar{\lambda})\right]-\sum_{\mathbf{S}}\left[P(\mathbf{S}|\mathbf{O},\lambda)\log P(\mathbf{S}|\mathbf{O},\bar{\lambda})\right]$$

The EM Algorithm

- Algorithm (Cont.)
  - We can thus express $\log P(\mathbf{O}|\bar{\lambda})$ as follows

$$\begin{aligned}
&\log P(\mathbf{O}|\bar{\lambda})\\
&=\sum_{\mathbf{S}}\left[P(\mathbf{S}|\mathbf{O},\lambda)\log P(\mathbf{O},\mathbf{S}|\bar{\lambda})\right]-\sum_{\mathbf{S}}\left[P(\mathbf{S}|\mathbf{O},\lambda)\log P(\mathbf{S}|\mathbf{O},\bar{\lambda})\right]\\
&=Q(\lambda,\bar{\lambda})-H(\lambda,\bar{\lambda})
\end{aligned}$$

where

$$Q(\lambda,\bar{\lambda})=\sum_{\mathbf{S}}\left[P(\mathbf{S}|\mathbf{O},\lambda)\log P(\mathbf{O},\mathbf{S}|\bar{\lambda})\right]$$

$$H(\lambda,\bar{\lambda})=\sum_{\mathbf{S}}\left[P(\mathbf{S}|\mathbf{O},\lambda)\log P(\mathbf{S}|\mathbf{O},\bar{\lambda})\right]$$

  - We want $\log P(\mathbf{O}|\bar{\lambda})\geq\log P(\mathbf{O}|\lambda)$

$$\begin{aligned}
&\log P(\mathbf{O}|\bar{\lambda})-\log P(\mathbf{O}|\lambda)\\
&=\left[Q(\lambda,\bar{\lambda})-H(\lambda,\bar{\lambda})\right]-\left[Q(\lambda,\lambda)-H(\lambda,\lambda)\right]\\
&=Q(\lambda,\bar{\lambda})-Q(\lambda,\lambda)\cancel{-H(\lambda,\bar{\lambda})+H(\lambda,\lambda)}
\end{aligned}$$

The EM Algorithm

- $-H(\lambda,\bar{\lambda})+H(\lambda,\lambda)$ has the following property

$$
\begin{aligned}
&-H(\lambda,\bar{\lambda})+H(\lambda,\lambda)\\
&=\boxed{-\sum_{\mathbf{S}}\left[P(\mathbf{S}|\mathbf{O},\lambda)\log\frac{P(\mathbf{S}|\mathbf{O},\bar{\lambda})}{P(\mathbf{S}|\mathbf{O},\lambda)}\right]}\\
&\geq\sum_{\mathbf{S}}\left[P(\mathbf{S}|\mathbf{O},\lambda)\left(1-\frac{P(\mathbf{S}|\mathbf{O},\bar{\lambda})}{P(\mathbf{S}|\mathbf{O},\lambda)}\right)\right] \quad (\because \log x\leq x-1)\\
&=\sum_{\mathbf{S}}\left[P(\mathbf{S}|\mathbf{O},\lambda)-P(\mathbf{S}|\mathbf{O},\bar{\lambda})\right]\\
&=\ 0\\
&\therefore\ -H(\lambda,\bar{\lambda})+H(\lambda,\lambda)\geq 0
\end{aligned}
$$

Kullbuack-Leibler (KL) distance

Jensen's inequality

– Therefore, for maximizing $\log P(\mathbf{O}|\bar{\lambda})$, we only need to maximize the *Q*-function (auxiliary function)

$$
Q(\lambda,\bar{\lambda})=\sum_{\mathbf{S}}\left[P(\mathbf{S}|\mathbf{O},\lambda)\log P(\mathbf{O},\mathbf{S}|\bar{\lambda})\right]
$$

Expectation of the complete data log likelihood with respect to the latent state sequences

EM Applied to Discrete HMM Training

- Apply EM algorithm to iteratively refine the HMM parameter vector $\boldsymbol{\lambda} = (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{\pi})$
  - By maximizing the auxiliary function

$$Q(\lambda, \bar{\lambda}) = \sum_{\boldsymbol{S}} \left[P(\boldsymbol{S} | \boldsymbol{O}, \lambda) log\ P(\boldsymbol{O}, \boldsymbol{S} | \bar{\lambda}) \right]$$

$$= \sum_{\boldsymbol{S}} \left[\frac{P(\boldsymbol{O}, \boldsymbol{S} | \lambda)}{P(\boldsymbol{O} | \lambda)} log\ P(\boldsymbol{O}, \boldsymbol{S} | \bar{\lambda}) \right]$$

  - Where $P(\boldsymbol{O}, \boldsymbol{S} | \lambda)$ and $P(\boldsymbol{O}, \boldsymbol{S} | \bar{\lambda})$ can be expressed as

$$P(\boldsymbol{O}, \boldsymbol{S} | \lambda) = \pi_{s_1} \left[\prod_{t=1}^{T-1} a_{s_t s_{t+1}} \right] \left[\prod_{t=1}^{T} b_{s_t}(\boldsymbol{o}_t) \right]$$

$$\log P(\boldsymbol{O}, \boldsymbol{S} | \lambda) = \log \pi_{s_1} + \sum_{t=1}^{T-1} \log a_{s_t s_{t+1}} + \sum_{t=1}^{T} \log b_{s_t}(\boldsymbol{o}_t)$$

$$\log P(\boldsymbol{O}, \boldsymbol{S} | \bar{\lambda}) = \log \bar{\pi}_{s_1} + \sum_{t=1}^{T-1} \log \bar{a}_{s_t s_{t+1}} + \sum_{t=1}^{T} \log \bar{b}_{s_t}(\boldsymbol{o}_t)$$

EM Applied to Discrete HMM Training

- Rewrite the auxiliary function as

$$Q\left(\lambda,\bar{\lambda}\right)=Q_{\boldsymbol{\pi}}\left(\lambda,\bar{\pi}\right)+Q_{\boldsymbol{a}}\left(\lambda,\bar{\boldsymbol{a}}\right)+Q_{\boldsymbol{b}}\left(\lambda,\bar{\boldsymbol{b}}\right)$$

$$Q_{\boldsymbol{\pi}}\left(\lambda,\bar{\pi}\right)=\sum_{\text{all }\boldsymbol{S}}\left[\frac{P\left(\boldsymbol{O},\boldsymbol{S}|\lambda\right)}{P\left(\boldsymbol{O}|\lambda\right)}\log\ \bar{\pi}_{s_1}\right]\overset{?}{=}\sum_{i=1}^{N}\left[\overbrace{\frac{P\left(\boldsymbol{O},s_1=i|\lambda\right)}{P\left(\boldsymbol{O}|\lambda\right)}}^{w_i}\log\ \overbrace{\bar{\pi}_i}^{y_i}\right]$$

$$Q_{\boldsymbol{a}}\left(\lambda,\bar{\boldsymbol{a}}\right)=\sum_{\text{all }\boldsymbol{S}}\left[\frac{P\left(\boldsymbol{O},\boldsymbol{S}|\lambda\right)}{P\left(\boldsymbol{O}|\lambda\right)}\sum_{t=1}^{T-1}\log\ \bar{a}_{s_t s_{t+1}}\right]\overset{?}{=}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T-1}\left[\frac{P\left(\boldsymbol{O},s_t=i,s_{t+1}=j|\lambda\right)}{P\left(\boldsymbol{O}|\lambda\right)}\log\ \bar{a}_{ij}\right]$$

$$Q_{\boldsymbol{b}}\left(\lambda,\bar{\boldsymbol{b}}\right)=\sum_{\text{all }\boldsymbol{S}}\left[\frac{P\left(\boldsymbol{O},\boldsymbol{S}|\lambda\right)}{P\left(\boldsymbol{O}|\lambda\right)}\sum_{t=1}^{T}\log\ \bar{b}_{s_t}(k)\right]\overset{?}{=}\sum_{j=1}^{N}\sum_{k}\sum_{t\in o_t=v_k}\left[\frac{P\left(\boldsymbol{O},s_t=j|\lambda\right)}{P\left(\boldsymbol{O}|\lambda\right)}\log\ \bar{b}_j(k)\right]$$

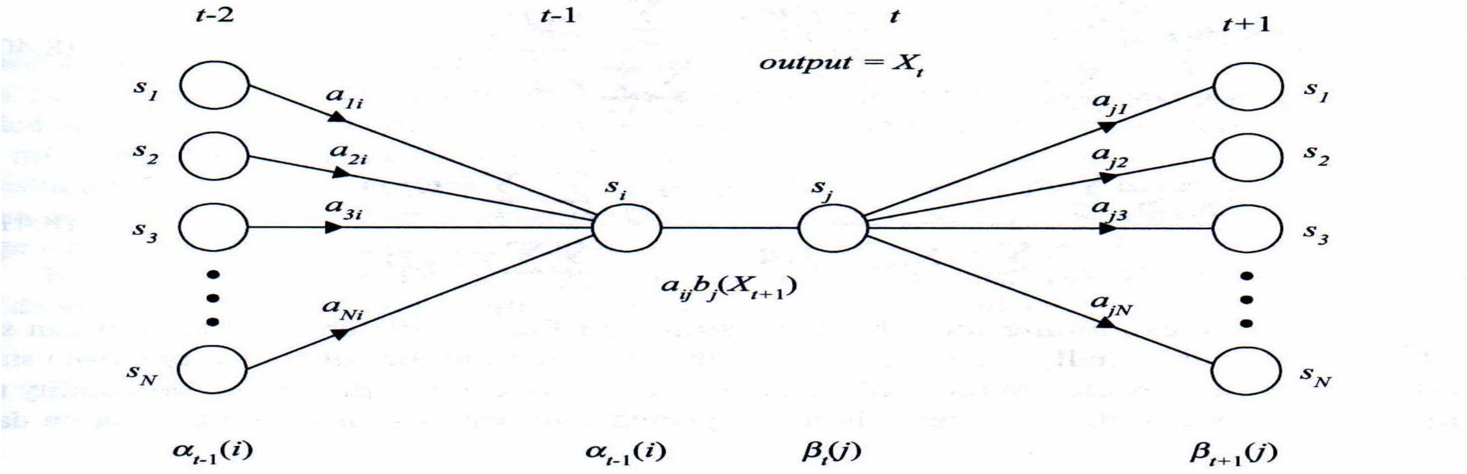


Figure 8.7 Illustration of the operations required for the computation of $\gamma_t(i,j)$, which is the probability of taking the transition from state i to state j at time t.

EM Applied to Discrete HMM Training

- The auxiliary function contains three independent terms, π_i , a_{ij} and $b_j(k)$
  - Can be maximized individually
  - All of the same form

$$F(\mathbf{y}) = g(y_1, y_2, \ldots, y_N) = \sum_{j=1}^{N} w_j \log y_j, \quad \text{where } \sum_{j=1}^{N} y_j = 1, \text{ and } y_j \geq 0$$

$$F(\mathbf{y}) \text{ has maximum value when : } y_j = \frac{w_j}{\sum_{j=1}^{N} w_j}$$

EM Applied to Discrete HMM Training

- **Proof**: Apply Lagrange Multiplier

By applying Lagrange Multiplier ℓ

Suppose that $F = \sum_{j=1}^{N} w_j \log y_j = \sum_{j=1}^{N} w_j \log y_j + \ell\left(\sum_{j=1}^{N} y_j - 1\right)$

Constraint

$$\frac{\partial F}{\partial y_j} = \frac{w_j}{y_j} + \ell = 0 \Rightarrow \ell = -\frac{w_j}{y_j} \ \forall\ j$$

$$\ell \sum_{j=1}^{N} y_j = -\sum_{j=1}^{N} w_j \Rightarrow \ell = -\sum_{j=1}^{N} w_j$$

$$\therefore y_j = \frac{w_j}{\sum_{j=1}^{N} w_j}$$

EM Applied to Discrete HMM Training

- The new model parameter set $\bar{\lambda} = \left(\bar{\boldsymbol{\pi}}, \bar{\boldsymbol{A}}, \bar{\boldsymbol{B}}\right)$ can be expressed as:

$$\overline{\pi}_i = \frac{P\left(\boldsymbol{O}, s_1 = i \middle| \lambda\right)}{P\left(\boldsymbol{O} \middle| \lambda\right)} = \gamma_1(i)$$

$$\overline{a}_{ij} = \frac{\sum_{t=1}^{T-1} P\left(\boldsymbol{O}, s_t = i, s_{t+1} = j \middle| \lambda\right)}{\sum_{t=1}^{T-1} P\left(\boldsymbol{O}, s_t = i \middle| \lambda\right)} = \frac{\sum_{t=1}^{T-1} \xi_t(i, j)}{\sum_{t=1}^{T-1} \gamma_t(i)}$$

$$\overline{b}_i(k) = \frac{\sum_{\substack{t=1 \\ \text{s.t. } o_t = v_k}}^{T} P\left(\boldsymbol{O}, s_t = i \middle| \lambda\right)}{\sum_{t=1}^{T} P\left(\boldsymbol{O}, s_t = i \middle| \lambda\right)} = \frac{\sum_{\substack{t=1 \\ \text{s.t. } o_t = v_k}}^{T} \gamma_t(i)}{\sum_{t=1}^{T} \gamma_t(i)}$$

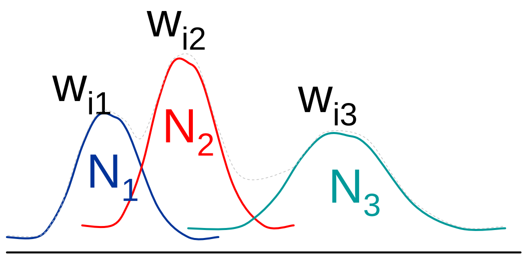
EM Applied to Continuous HMM Training

- Continuous HMM: the state observation does not come from a finite set, but from a continuous space
  - The difference between the discrete and continuous HMM lies in a different form of state output probability
  - Discrete HMM requires the quantization procedure to map observation vectors from the continuous space to the discrete space
- Continuous Mixture HMM
  - The state observation distribution of HMM is modeled by multivariate Gaussian mixture density functions (*M* mixtures)

$$b_j(\boldsymbol{o}) = \sum_{k=1}^{M} c_{jk} b_{jk}(\boldsymbol{o})$$

$$= \sum_{k=1}^{M} c_{jk} N(\boldsymbol{o};\boldsymbol{\mu}_{jk},\boldsymbol{\Sigma}_{jk}) = \sum_{k=1}^{M} c_{jk}\left(\frac{1}{\left(\sqrt{2\pi}\right)^{L}\left|\boldsymbol{\Sigma}_{jk}\right|^{1/2}} \exp\left(-\frac{1}{2}(\boldsymbol{o}-\boldsymbol{\mu}_{jk})^{t}\boldsymbol{\Sigma}_{jk}^{-1}(\boldsymbol{o}-\boldsymbol{\mu}_{jk})\right)\right)$$

$$\sum_{k=1}^{M} c_{jk} = 1$$



Distribution for State *i*

EM Applied to Continuous HMM Training

- Express $b_j(\boldsymbol{o})$ with respect to each single mixture component $b_{jk}(\boldsymbol{o})$

Note:

$$\prod_{t=1}^{T}\left(\sum_{k_t=1}^{M} a_{tk_t}\right)$$

$$=(a_{11}+a_{12}+...+a_{1M})(a_{21}+a_{22}+...+a_{2M})...(a_{T1}+a_{T2}+...+a_{TM})$$

$$=\sum_{k_1=1}^{M}\sum_{k_2=1}^{M}...\sum_{k_T=1}^{M}\prod_{t=1}^{T} a_{tk_t}$$

$$P(\boldsymbol{O},\boldsymbol{S}|\lambda)=\pi_{s_1}\left\{\prod_{t=1}^{T-1} a_{s_t s_{t+1}}\right\}\left\{\prod_{t=1}^{T} b_{s_t}(\boldsymbol{o}_t)\right\}$$

$$=\pi_{s_1}\left\{\prod_{t=1}^{T-1} a_{s_t s_{t+1}}\right\}\left\{\sum_{k_1=1}^{M}\sum_{k_2=1}^{M}\cdots\sum_{k_T=1}^{M}\boxed{\prod_{t=1}^{T}\left[c_{s_t k_t} b_{s_t k_t}(\boldsymbol{o}_t)\right]}\right\}$$

$$P(\boldsymbol{O},\boldsymbol{S},\boldsymbol{K}|\lambda)=\pi_{s_1}\left\{\prod_{t=1}^{T-1} a_{s_t s_{t+1}}\right\}\left\{\prod_{t=1}^{T}\left[c_{s_t k_t} b_{s_t k_t}(\boldsymbol{o}_t)\right]\right\}$$

$\boldsymbol{K}$: one of the possible mixture component sequence along with the state sequence $\boldsymbol{S}$

$$P(\boldsymbol{O}|\lambda)=\sum_{\boldsymbol{S}}\sum_{\boldsymbol{K}} P(\boldsymbol{O},\boldsymbol{S},\boldsymbol{K}|\lambda)$$

EM Applied to Continuous HMM Training

- Therefore, an auxiliary function for the EM algorithm can be written as:

$$Q\left(\lambda,\bar{\lambda}\right)=\sum_{\boldsymbol{S}}\sum_{\boldsymbol{K}}\left[P\left(\boldsymbol{S},\boldsymbol{K}\middle|\boldsymbol{O},\lambda\right)\log P\left(\boldsymbol{O},\boldsymbol{S},\boldsymbol{K}\middle|\bar{\lambda}\right)\right]$$

$$=\sum_{\boldsymbol{S}}\sum_{\boldsymbol{K}}\left[\frac{P\left(\boldsymbol{O},\boldsymbol{S},\boldsymbol{K}\middle|\lambda\right)}{P\left(\boldsymbol{O}\middle|\lambda\right)}\log P\left(\boldsymbol{O},\boldsymbol{S},\boldsymbol{K}\middle|\bar{\lambda}\right)\right]$$

$$\log P\left(\boldsymbol{O},\boldsymbol{S},\boldsymbol{K}\middle|\bar{\lambda}\right)=\log\bar{\pi}_{s_1}+\sum_{t=1}^{T-1}\log\bar{a}_{s_t s_{t+1}}+\sum_{t=1}^{T}\log\bar{b}_{s_t k_t}\left(\boldsymbol{o}_t\right)+\sum_{t=1}^{T}\log\bar{c}_{s_t k_t}$$

$$\Big\Downarrow$$

$$Q\left(\lambda,\bar{\lambda}\right)=Q_{\boldsymbol{\pi}}\left(\lambda,\bar{\pi}\right)+Q_{\boldsymbol{a}}\left(\lambda,\bar{\boldsymbol{a}}\right)+Q_{\boldsymbol{b}}\left(\lambda,\bar{\boldsymbol{b}}\right)+Q_{c}\left(\lambda,\bar{c}\right)$$

initial probabilities

state transition probabilities

Gaussian density functions

mixture components

EM Applied to Continuous HMM Training

- The only difference we have when compared with Discrete HMM training

$$Q_{\boldsymbol{b}}\left(\lambda,\overline{\boldsymbol{b}}\right)=\sum_{t=1}^{T}\left\{\left[\sum_{j=1}^{N}\sum_{k=1}^{M}\overset{\gamma_t(j,k)}{\boxed{P\left(s_t=j,k_t=k\middle|\boldsymbol{O},\boldsymbol{\lambda}\right)}}\right]\log\overline{b}_{jk}\left(\boldsymbol{o}_t\right)\right\}$$

$$Q_{c}\left(\lambda,\overline{\boldsymbol{c}}\right)=\sum_{t=1}^{T}\left\{\left[\sum_{j=1}^{N}\sum_{k=1}^{M}P\left(s_t=j,k_t=k\middle|\boldsymbol{O},\boldsymbol{\lambda}\right)\right]\log\overline{c}_{jk}\left(\boldsymbol{o}_t\right)\right\}$$

EM Applied to Continuous HMM Training

Let $\gamma_t(j,k) = \sum_{k=1}^{M} P\left(s_t = j, k_t = k \middle| \boldsymbol{O}, \boldsymbol{\lambda}\right)$

$$\overline{b}_{jk}(\boldsymbol{o}_t) = N\left(\boldsymbol{o}_t ; \overline{\boldsymbol{\mu}}_{jk}, \overline{\boldsymbol{\Sigma}}_{jk}\right) = \frac{1}{(2\pi)^{L/2}\left|\overline{\boldsymbol{\Sigma}}_{jk}\right|^{1/2}} \exp\left(-\frac{1}{2}\left(\boldsymbol{o}_t - \overline{\boldsymbol{\mu}}_{jk}\right)^t \overline{\boldsymbol{\Sigma}}_{jk}^{-1}\left(\boldsymbol{o}_t - \overline{\boldsymbol{\mu}}_{jk}\right)\right)$$

$$\log \overline{b}_{jk}(\boldsymbol{o}_t) = -L/2 \cdot \log(2\pi) + 1/2 \cdot \log\left|\overline{\boldsymbol{\Sigma}}_{jk}^{-1}\right| - \left(\frac{1}{2}\left(\boldsymbol{o}_t - \overline{\boldsymbol{\mu}}_{jk}\right)^t \overline{\boldsymbol{\Sigma}}_{jk}^{-1}\left(\boldsymbol{o}_t - \overline{\boldsymbol{\mu}}_{jk}\right)\right)$$

$$\frac{\partial \log \overline{b}_{jk}(\boldsymbol{o}_t)}{\partial \overline{\boldsymbol{\mu}}_{jk}} = \overline{\boldsymbol{\Sigma}}_{jk}^{-1}\left(\boldsymbol{o}_t - \overline{\boldsymbol{\mu}}_{jk}\right)$$

$$\frac{d\,(\boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{C}\boldsymbol{x})}{d\boldsymbol{x}} = \left(\boldsymbol{C} + \boldsymbol{C}^{\boldsymbol{T}}\right)\boldsymbol{x}$$

and Σ_{jk}^{-1} is symmetric here

$$\frac{\partial Q_{\boldsymbol{b}}\left(\boldsymbol{\lambda}, \overline{\boldsymbol{b}}\right)}{\partial \overline{\boldsymbol{\mu}}_{jk}} = \frac{\partial \sum_{t=1}^{T}\left\{\left[\sum_{j=1}^{N}\sum_{k=1}^{M} \gamma_t(j,k) \log \overline{b}_{jk}(\boldsymbol{o}_t)\right]\right\}}{\partial \overline{\boldsymbol{\mu}}_{jk}}$$

$$\Rightarrow \sum_{t=1}^{T}\left\{\gamma_t(j,k)\overline{\boldsymbol{\Sigma}}_{jk}^{-1}\left(\boldsymbol{o}_t - \overline{\boldsymbol{\mu}}_{jk}\right)\right\} = 0$$

$$\Rightarrow \overline{\boldsymbol{\mu}}_{jk} = \frac{\sum_{t=1}^{T}\left[\gamma_t(j,k) \cdot \boldsymbol{o}_t\right]}{\sum_{t=1}^{T}\gamma_t(j,k)}$$

EM Applied to Continuous HMM Training

$$\log \bar{b}_{jk}(\boldsymbol{o}_t) = -L/2 \cdot \log(2\pi) - 1/2 \cdot \log\left|\bar{\boldsymbol{\Sigma}}_{jk}\right| - \left(\frac{1}{2}\left(\boldsymbol{o}_t - \bar{\boldsymbol{\mu}}_{jk}\right)^t \bar{\boldsymbol{\Sigma}}_{jk}^{-1}\left(\boldsymbol{o}_t - \bar{\boldsymbol{\mu}}_{jk}\right)\right)$$

$$\frac{\partial \log \bar{b}_{jk}(\boldsymbol{o}_t)}{\partial\left(\bar{\boldsymbol{\Sigma}}_{jk}\right)} = -\left[\frac{1}{2}\cdot\cancel{\left|\bar{\boldsymbol{\Sigma}}_{jk}\right|^{-1}\cdot\left|\bar{\boldsymbol{\Sigma}}_{jk}\right|}\cdot\bar{\boldsymbol{\Sigma}}_{jk}^{-1} - \left(\bar{\boldsymbol{\Sigma}}_{jk}^{-1}\frac{1}{2}\left(\boldsymbol{o}_t - \bar{\boldsymbol{\mu}}_{jk}\right)\left(\boldsymbol{o}_t - \bar{\boldsymbol{\mu}}_{jk}\right)^t\bar{\boldsymbol{\Sigma}}_{jk}^{-1}\right)\right]$$

$$= -\frac{1}{2}\cdot\left[\bar{\boldsymbol{\Sigma}}_{jk}^{-1} - \bar{\boldsymbol{\Sigma}}_{jk}^{-1}\left(\boldsymbol{o}_t - \bar{\boldsymbol{\mu}}_{jk}\right)\left(\boldsymbol{o}_t - \bar{\boldsymbol{\mu}}_{jk}\right)^t\bar{\boldsymbol{\Sigma}}_{jk}^{-1}\right]$$

$$\frac{d\left(\boldsymbol{a}^T\boldsymbol{X}^{-1}\boldsymbol{b}\right)}{d\boldsymbol{X}} = -\boldsymbol{X}^T\boldsymbol{a}\boldsymbol{b}^T\boldsymbol{X}^T$$

$$\frac{d\left[\det(\boldsymbol{X})\right]}{d\boldsymbol{X}} = \det(\boldsymbol{X})\cdot\boldsymbol{X}^{-T}$$

and Σ_{jk} is symmetric here

$$\frac{\partial Q_{\boldsymbol{b}}\left(\lambda, \bar{\boldsymbol{b}}\right)}{\partial\left(\bar{\boldsymbol{\Sigma}}_{jk}\right)} = \frac{\partial\sum_{t=1}^{T}\left\{\left[\sum_{j=1}^{N}\sum_{k=1}^{M}\gamma_t(j,k)\log \bar{b}_{jk}(\boldsymbol{o}_t)\right]\right\}}{\partial\left(\bar{\boldsymbol{\Sigma}}_{jk}^{-1}\right)}$$

$$\Rightarrow \sum_{t=1}^{T}\left\{\gamma_t(j,k)\left(-\frac{1}{2}\right)\cdot\left[\bar{\boldsymbol{\Sigma}}_{jk}^{-1} - \bar{\boldsymbol{\Sigma}}_{jk}^{-1}\left(\boldsymbol{o}_t - \bar{\boldsymbol{\mu}}_{jk}\right)\left(\boldsymbol{o}_t - \bar{\boldsymbol{\mu}}_{jk}\right)^t\bar{\boldsymbol{\Sigma}}_{jk}^{-1}\right]\right\} = 0$$

$$\Rightarrow \sum_{t=1}^{T}\gamma_t(j,k)\bar{\boldsymbol{\Sigma}}_{jk}^{-1} = \sum_{t=1}^{T}\gamma_t(j,k)\bar{\boldsymbol{\Sigma}}_{jk}^{-1}\left(\boldsymbol{o}_t - \bar{\boldsymbol{\mu}}_{jk}\right)\left(\boldsymbol{o}_t - \bar{\boldsymbol{\mu}}_{jk}\right)^t\bar{\boldsymbol{\Sigma}}_{jk}^{-1}$$

$$\Rightarrow \sum_{t=1}^{T}\gamma_t(j,k)\bar{\boldsymbol{\Sigma}}_{jk}\bar{\boldsymbol{\Sigma}}_{jk}^{-1}\bar{\boldsymbol{\Sigma}}_{jk} = \sum_{t=1}^{T}\gamma_t(j,k)\bar{\boldsymbol{\Sigma}}_{jk}\bar{\boldsymbol{\Sigma}}_{jk}^{-1}\left(\boldsymbol{o}_t - \bar{\boldsymbol{\mu}}_{jk}\right)\left(\boldsymbol{o}_t - \bar{\boldsymbol{\mu}}_{jk}\right)^t\bar{\boldsymbol{\Sigma}}_{jk}^{-1}\bar{\boldsymbol{\Sigma}}_{jk}$$

$$\Rightarrow \bar{\boldsymbol{\Sigma}}_{jk} = \frac{\sum_{t=1}^{T}\left[\gamma_t(j,k)\cdot\left(\boldsymbol{o}_t - \bar{\boldsymbol{\mu}}_{jk}\right)\left(\boldsymbol{o}_t - \bar{\boldsymbol{\mu}}_{jk}\right)^t\right]}{\sum_{t=1}^{T}\gamma_t(j,k)}$$

EM Applied to Continuous HMM Training

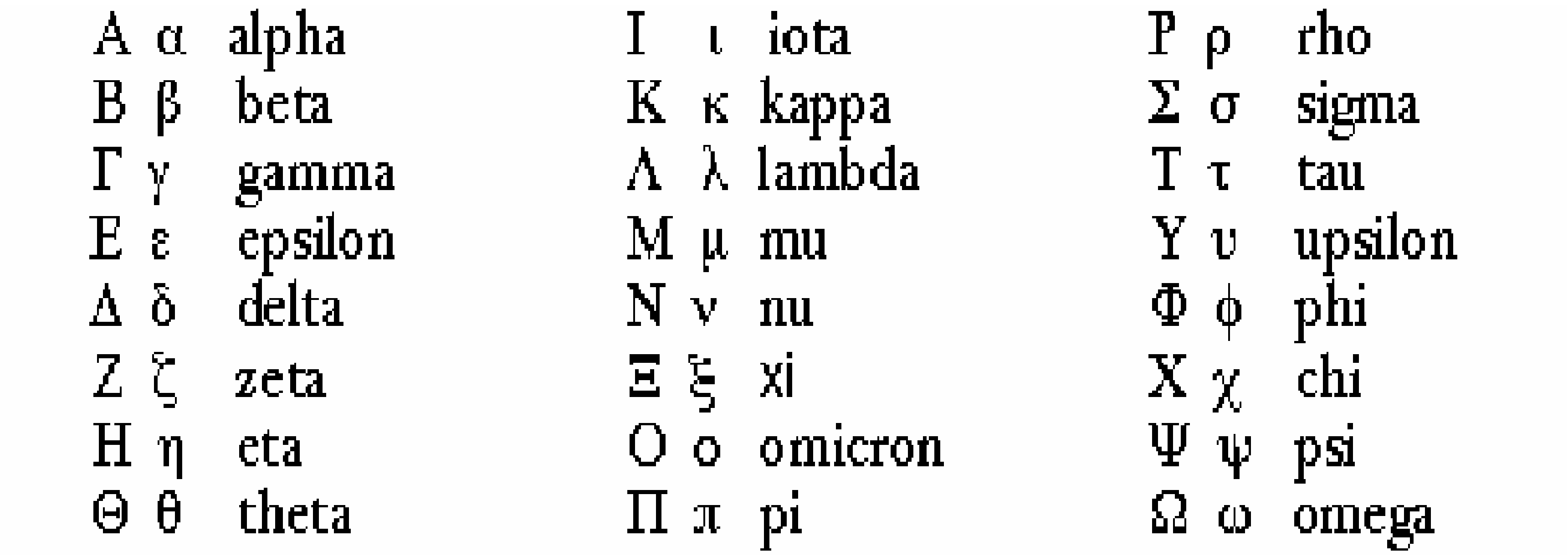
- The new model parameter set for each mixture component and mixture weight can be expressed as:

$$\overline{\boldsymbol{\mu}}_{jk} = \frac{\sum_{t=1}^{T}\left[\frac{P(\boldsymbol{O}, s_t = j, k_t = k|\boldsymbol{\lambda})}{P(\boldsymbol{O}|\boldsymbol{\lambda})}\boldsymbol{o}_t\right]}{\sum_{t=1}^{T}\frac{P(\boldsymbol{O}, s_t = j, k_t = k|\boldsymbol{\lambda})}{P(\boldsymbol{O}|\boldsymbol{\lambda})}} = \frac{\sum_{t=1}^{T}\left[\gamma_t(j,k)\boldsymbol{o}_t\right]}{\sum_{t=1}^{T}\gamma_t(j,k)}$$

$$\overline{\boldsymbol{\Sigma}}_{jk} = \frac{\sum_{t=1}^{T}\left[\frac{P(\boldsymbol{O}, s_t = j, k_t = k|\boldsymbol{\lambda})}{P(\boldsymbol{O}|\boldsymbol{\lambda})}(\boldsymbol{o}_t - \overline{\boldsymbol{\mu}}_{jk})(\boldsymbol{o}_t - \overline{\boldsymbol{\mu}}_{jk})^t\right]}{\sum_{t=1}^{T}\frac{P(\boldsymbol{O}, s_t = j, k_t = k|\boldsymbol{\lambda})}{P(\boldsymbol{O}|\boldsymbol{\lambda})}} = \frac{\sum_{t=1}^{T}\left[\gamma_t(j,k)(\boldsymbol{o}_t - \overline{\boldsymbol{\mu}}_{jk})(\boldsymbol{o}_t - \overline{\boldsymbol{\mu}}_{jk})^t\right]}{\sum_{t=1}^{T}\gamma_t(j,k)}$$

$$\overline{c}_{jk} = \frac{\sum_{t=1}^{T}\gamma_t(j,k)}{\sum_{t=1}^{T}\sum_{k=1}^{M}\gamma_t(j,k)}$$

Symbols for Mathematical Operations



Α	α	alpha	Ι	ι	iota	Ρ	ρ	rho
Β	β	beta	Κ	κ	kappa	Σ	σ	sigma
Γ	γ	gamma	Λ	λ	lambda	Τ	τ	tau
Ε	ε	epsilon	Μ	μ	mu	Υ	υ	upsilon
Δ	δ	delta	Ν	ν	nu	Φ	φ	phi
Ζ	ζ	zeta	Ξ	ξ	xi	Χ	χ	chi
Η	η	eta	Ο	ο	omicron	Ψ	ψ	psi
Θ	θ	theta	Π	π	pi	Ω	ω	omega