Digital Signal Processing

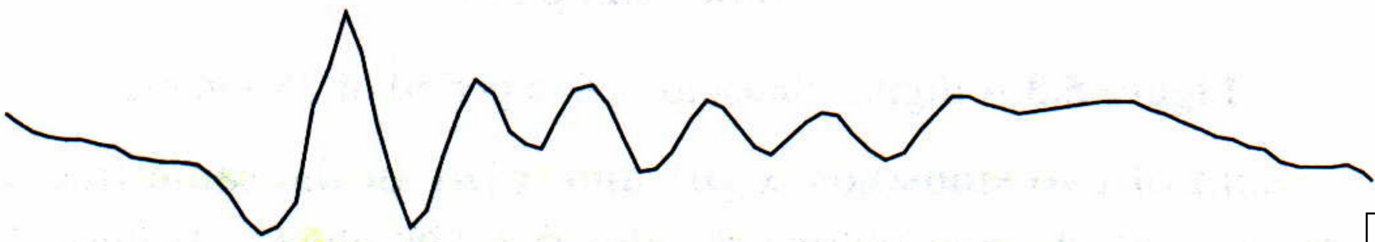
Berlin Chen 2004

References:

1. X. Huang et. al., Spoken Language Processing, Chapters 5, 6
2. J. R. Deller et. al., Discrete-Time Processing of Speech Signals, Chapters 4-6
3. J. W. Picone, “Signal modeling techniques in speech recognition,” proceedings of the IEEE, September 1993, pp. 1215-1247

Analog Signal to Digital Signal

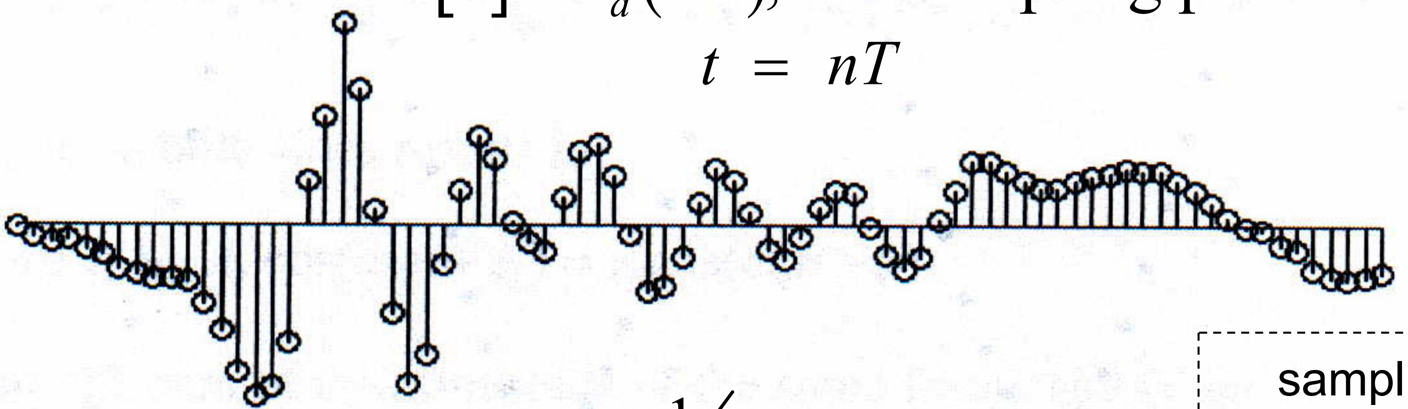
Analog Signal



Discrete-time Signal or Digital Signal

$$x[n] = x_a(nT), \quad T: \text{sampling period}$$

$$t = nT$$



$$F_s = 1/T \quad \text{sampling rate}$$

Digital Signal:
Discrete-time signal with discrete amplitude

sampling period=125 μs
=>sampling rate=8kHz

Analog Signal to Digital Signal (cont.)

Continuous-Time to Discrete-Time Conversion

Continuous-Time Signal

$x_a(t)$ → Sampling → $x_s(t)$ → Impulse Train To Sequence → $\hat{x}[n] \ (= x_a(nT))$ → $x[n]$

Discrete-Time Signal

switch

$$= x_a(t)s(t) = \sum_{n=-\infty}^{\infty} x_a(t)\delta(t-nT)$$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

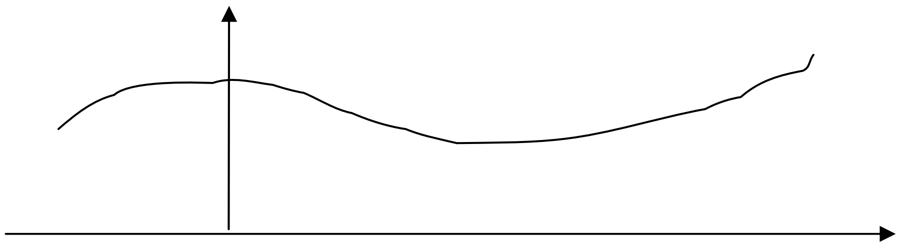
Periodic Impulse Train

$$= \sum_{n=-\infty}^{\infty} x_a(nT)\delta(t-nT) = \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)$$

$x_s(t)$ can be uniquely specified by $x[n]$

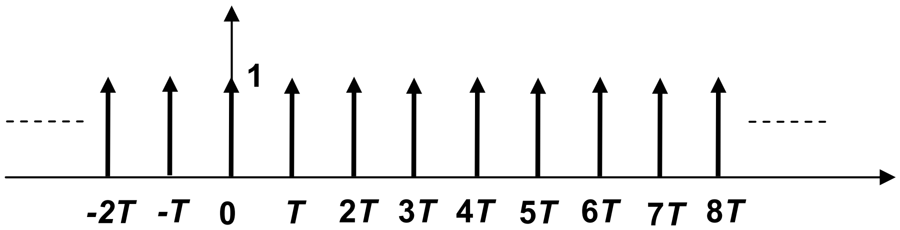
Digital Signal

$x_a(t)$



Discrete-time signal with discrete amplitude

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

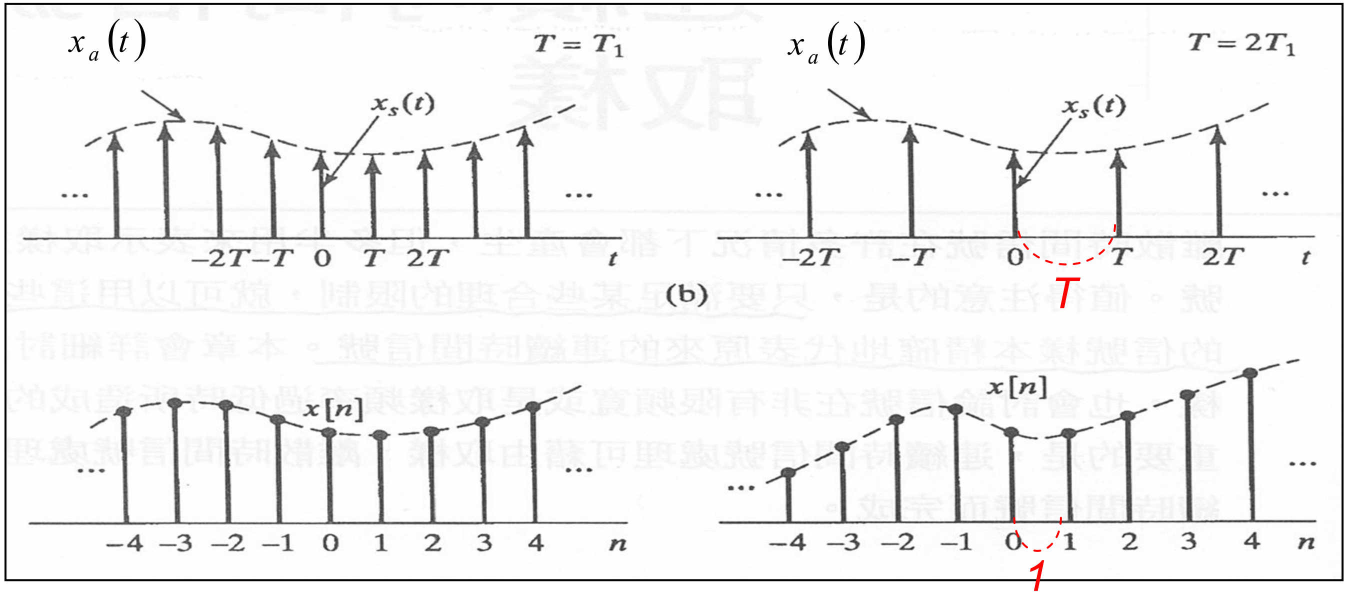


$$\delta(t) = 0, \quad \forall t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

Analog Signal to Digital Signal (cont.)

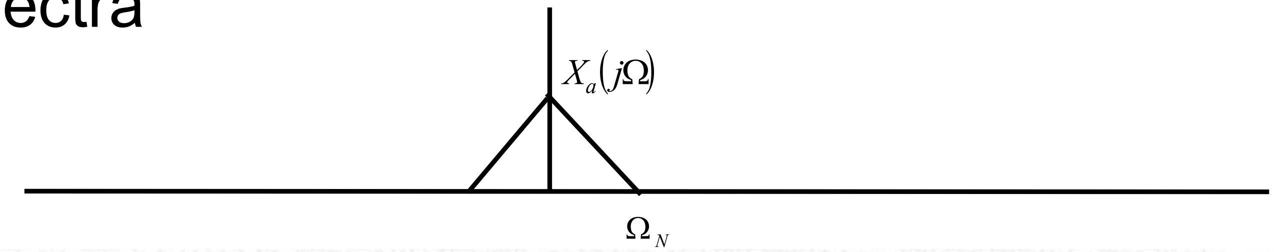
- A continuous signal sampled at different periods



$$
\begin{aligned}
& x_s(t) \\
& = x_a(t)s(t) = \sum_{n=-\infty}^{\infty} x_a(t)\delta(t-nT) \\
& = \sum_{n=-\infty}^{\infty} x_a(nT)\delta(t-nT) = \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)
\end{aligned}
$$

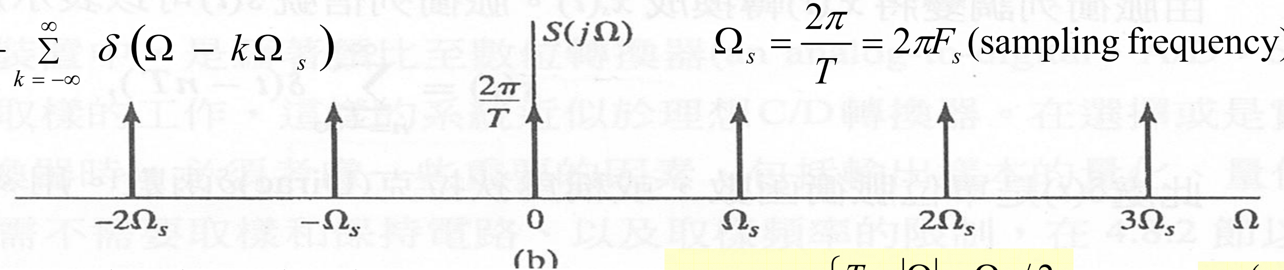
Analog Signal to Digital Signal (cont.)

- Spectra



$$S(j\Omega)=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\Omega-k\Omega_s)$$

$$\Omega_s=\frac{2\pi}{T}=2\pi F_s \text{ (sampling frequency)}$$

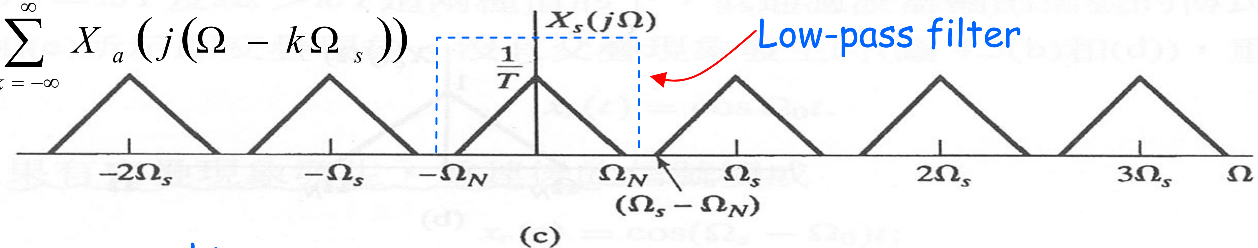


$$X_s(j\Omega)=\frac{1}{2\pi}X_a(j\Omega)*S(j\Omega)$$

$$R_{\Omega_s}(j\Omega)=\begin{cases}T & |\Omega|<\Omega_s/2\\ 0 & \text{otherwise}\end{cases} \Rightarrow X_a(j\Omega)=R_{\Omega_s}(j\Omega)X_p(j\Omega)$$

$$X_s(j\Omega)=\frac{1}{T}\sum_{k=-\infty}^{\infty}X_a(j(\Omega-k\Omega_s))$$

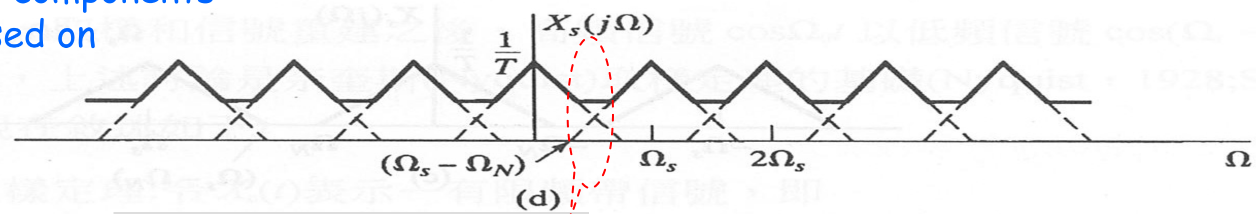
Low-pass filter



$$\Omega_N<\frac{1}{2}\Omega_s\left(=\frac{\pi}{T}\right)$$

$$\begin{Bmatrix}\because(\Omega_s-\Omega_N)>\Omega_N\\ \Rightarrow\Omega_s>2\Omega_N\end{Bmatrix}$$

high frequency components got superimposed on low frequency components



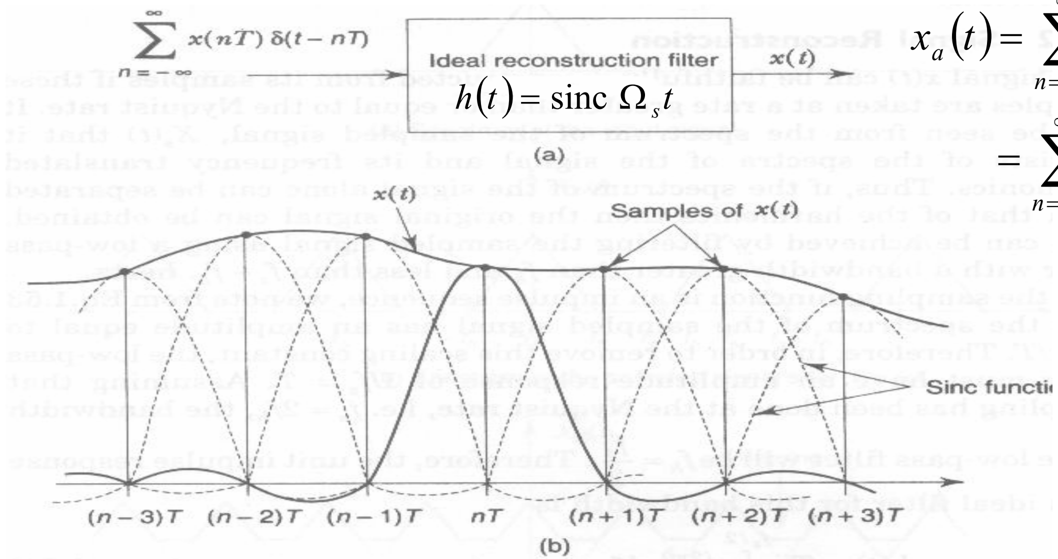
$$\Omega_N>\frac{1}{2}\Omega_s\left(=\frac{\pi}{T}\right)$$

$$\begin{Bmatrix}\because(\Omega_s-\Omega_N)<\Omega_N\\ \Rightarrow\Omega_s<2\Omega_N\end{Bmatrix}$$

aliasing distortion $\Rightarrow$ $X_a(j\Omega)$ can't be recovered from $X_p(j\Omega)$

Analog Signal to Digital Signal (cont.)

- To avoid aliasing (*overlapping*, *fold over*)
  - The sampling frequency should be greater than two times of frequency of the signal to be sampled $\rightarrow \ \Omega_s > 2\Omega_N$
  - (Nyquist) sampling theorem
- To reconstruct the original continuous signal
  - Filtered with a low pass filter with band limit Ω_s
    - Convolved in time domain



$$h(t) = \text{sinc}\ \Omega_s t$$

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT)h(t-nT)$$

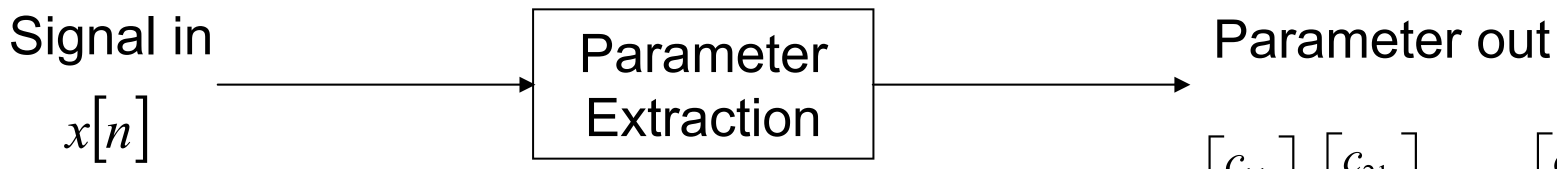
$$= \sum_{n=-\infty}^{\infty} x_a(nT)\text{sinc}\ \Omega_s(t-nT)$$

Two Main Approaches to Digital Signal Processing

- Filtering

Amplify or attenuate some frequency components of $x[n]$

- Parameter Extraction



$$\begin{bmatrix} c_{11} \\ c_{12} \\ \\ \\ c_{1m} \end{bmatrix} \begin{bmatrix} c_{21} \\ c_{22} \\ \\ \\ c_{2m} \end{bmatrix} \quad \begin{bmatrix} c_{L1} \\ c_{L2} \\ \\ \\ c_{Lm} \end{bmatrix}$$

e.g.:
1. Spectrum Estimation
2. Parameters for Recognition

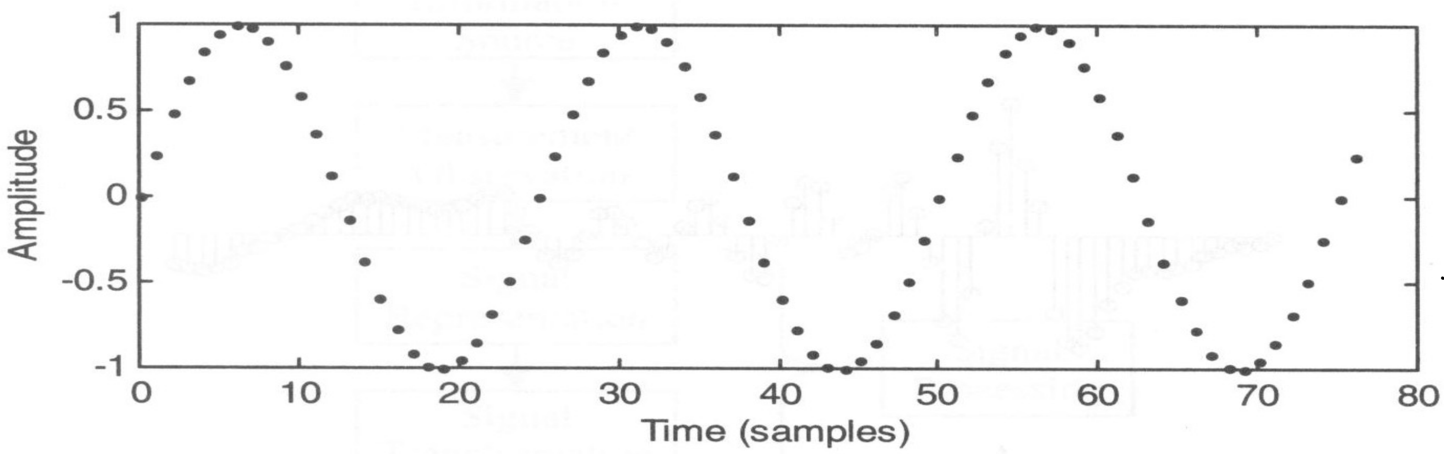
Sinusoid Signals

$$x[n] = A\cos(\omega n + \phi)$$

f : normalized frequency
$0 \le f \le 1$

- A : amplitude (振幅)
- ω : angular frequency (角頻率), $\omega = 2\pi f = \dfrac{2\pi}{\boxed{T}}$
- ϕ : phase (相角)

Period, represented by number of samples



$$x[n] = A\cos\left(\omega n - \frac{\pi}{2}\right)$$

$T = 25$ samples

Sinusoid Signals (cont.)

- $x[n]$ is periodic with a period of *N* (samples)

$$\Rightarrow x[n+N] = x[n]$$

$$\Rightarrow A\cos(\omega(n+N)+\phi) = A\cos(\omega n+\phi)$$

$$\Rightarrow \omega N = 2\pi$$

$$\Rightarrow \omega = \frac{2\pi}{N}$$

- Examples (sinusoid signals)
  - $x_1[n] = \cos(\pi n/4)$ is periodic with period *N*=8
  - $x_2[n] = \cos(3\pi n/8)$ is periodic with period *N*=16
  - $x_3[n] = \cos(n)$ is not periodic

Sinusoid Signals (cont.)

$$x_1[n] = \cos(\pi n / 4)$$

$$= \cos\left(\frac{\pi}{4} n\right) = \cos\left(\frac{\pi}{4}(n + N_1)\right) = \cos\left(\frac{\pi}{4} n + \frac{\pi}{4} N_1\right)$$

$$\Rightarrow \frac{\pi}{4} N_1 = 2\pi \cdot k \Rightarrow 8 \cdot k \qquad (N_1 \text{ and } k \text{ are positive integers})$$

$$\therefore N_1 = 8$$

$$x_2[n] = \cos(3\pi n / 8)$$

$$= \cos\left(\frac{3\pi}{8} \cdot n\right) = \cos\left(\frac{3\pi}{8}(n + N_2)\right) = \cos\left(\frac{3\pi}{8} \cdot n + \frac{3\pi}{8} \cdot N_2\right)$$

$$\Rightarrow \frac{3\pi}{8} \cdot N_2 = 2\pi \cdot k \Rightarrow N_2 = \frac{16}{3} k \qquad (N_2 \text{ and } k \text{ are positive numbers})$$

$$\therefore N_2 = 16$$

$$x_3[n] = \cos(n)$$

$$= \cos(1 \cdot n) = \cos(1 \cdot (n + N_3)) = \cos(n + N_3)$$

$$\Rightarrow N_3 = 2\pi \cdot k$$

$$\because N_3 \text{ and } k \text{ are positive integers}$$

$$\therefore N_3 \text{ doesn't exist !}$$

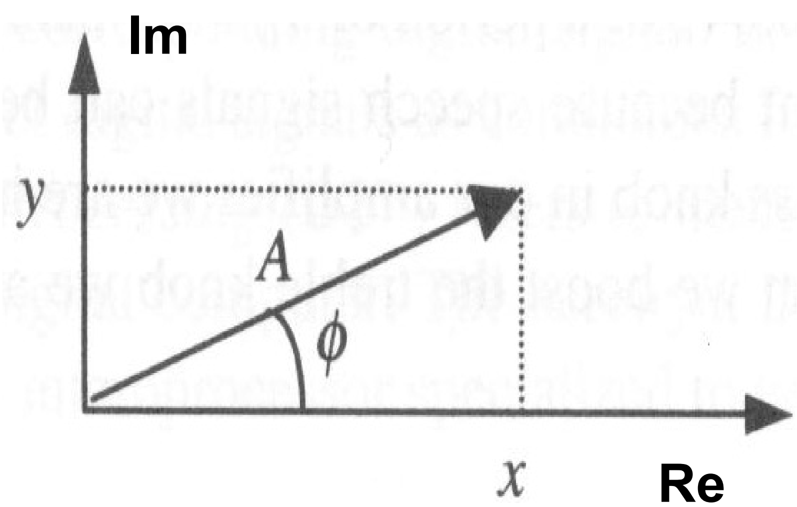
Sinusoid Signals (cont.)

- Complex Exponential Signal
  - Use **Euler's relation** to express complex numbers

$$z = x + jy$$

$$\Rightarrow z = Ae^{j\phi} = A(\cos\phi + j\sin\phi)$$

(A is a real number)



$$x = A\cos\phi$$

$$y = A\sin\phi$$

Sinusoid Signals (cont.)

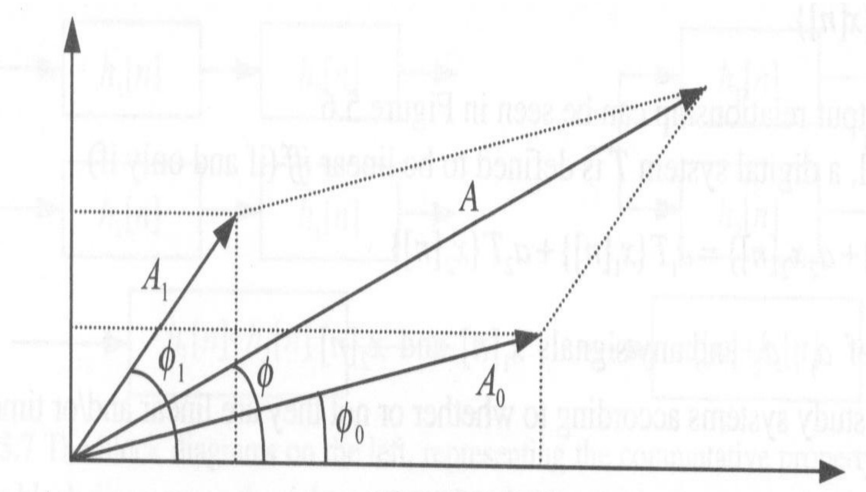
- A Sinusoid Signal

$$
\begin{aligned}
x[n] &= A\cos(\omega n + \phi) \\
&\boxed{= \operatorname{Re}\left\{Ae^{j(\omega n + \phi)}\right\}} \\
&= \operatorname{Re}\left\{Ae^{j\omega n}e^{j\phi}\right\}
\end{aligned}
$$

Sinusoid Signals (cont.)

- Sum of two complex exponential signals with same frequency

$$A_0 e^{j(\omega n + \phi_0)} + A_1 e^{j(\omega n + \phi_1)}$$
$$= e^{j\omega n}\left(A_0 e^{j\phi_0} + A_1 e^{j\phi_1}\right)$$
$$= e^{j\omega n} A e^{j\phi}$$
$$= A e^{j(\omega n + \phi)}$$



A, A_0 and A_1 are real numbers

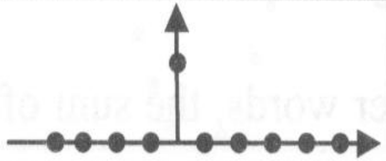
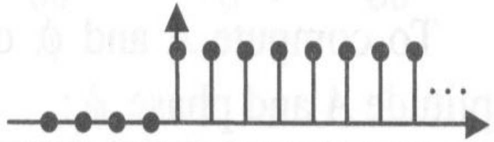
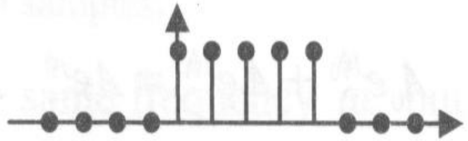
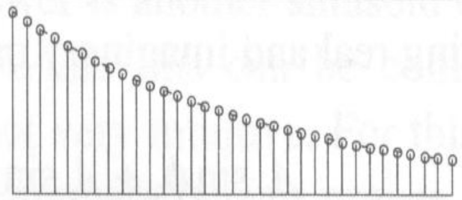
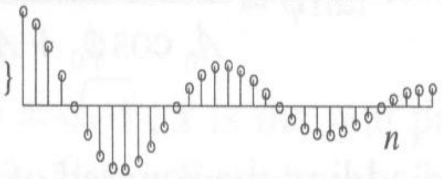
- When only the real part is considered

$$A_0 \cos(\omega n + \phi_0) + A_1 \cos(\omega n + \phi_1) = A \cos(\omega n + \phi)$$

- The sum of *N* sinusoids of the same frequency is another sinusoid of the same frequency

Some Digital Signals

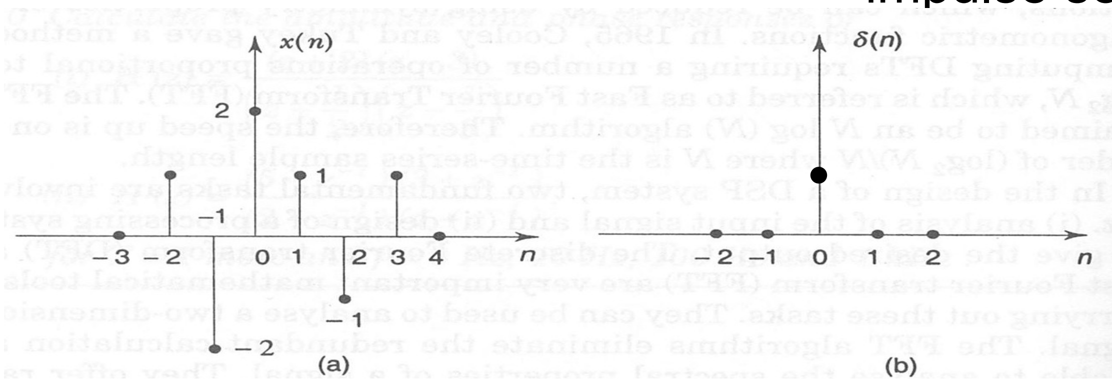
Table 5.1 Some useful digital signals: the Kronecker delta, unit step, rectangular signal, real exponential ($a<1$) and real part of a complex exponential ($r<1$).

Kronecker delta, or unit impulse	$\delta[n]=\begin{cases}1 & n=0\\ 0 & otherwise\end{cases}$	
Unit step	$u[n]=\begin{cases}1 & n\geq 0\\ 0 & n<0\end{cases}$	
Rectangular signal	$\mathrm{rect}_N[n]=\begin{cases}1 & 0\leq n<N\\ 0 & otherwise\end{cases}$	
Real exponential	$x[n]=a^n u[n]$	
Complex exponential	$x[n]=a^n u[n]=r^n e^{jn\omega_0}u[n]$ $=r^n(\cos n\omega_0+j\sin n\omega_0)u[n]$	

Some Digital Signals

- Any signal sequence $x[n]$ can be represented as a sum of *shift* and *scaled* unit impulse sequences (signals)

$$x[n] = \sum_{k=-\infty}^{\infty} \underbrace{\boxed{x[k]}}_{\text{scale/weighted}} \underbrace{\boxed{\delta[n-k]}}_{\text{Time-shifted unit impulse sequence}}$$



-

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] = \sum_{k=-2}^{3} x[k]\,\delta[n-k]$$

$$= x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3]$$

$$= (1)\delta[n+2] + (-2)\delta[n+1] + (2)\delta[n] + (1)\delta[n-1] + (-1)\delta[n-2] + (1)\delta[n-3]$$

Digital Systems

- A digital system *T* is a system that, given an input signal *x*[*n*], generates an output signal *y*[*n*]

$$y[n] = T\{x[n]\}$$

$$x[n] \longrightarrow \boxed{\quad T\{\ \}\quad} \longrightarrow y[n]$$

Properties of Digital Systems

- Linear
  - Linear combination of inputs maps to linear combination of outputs

$$T\{ax_1[n]+bx_2[n]\}=aT\{x_1[n]\}+bT\{x_2[n]\}$$

- Time-invariant (Time-shift)
  - A time shift of in the input by *m* samples give a shift in the output by *m* samples

$$y[n \pm m]=T\{x[n \pm m]\}, \quad \forall m$$

Properties of Digital Systems (cont.)

- Linear time-invariant (LTI)
  - The system output can be expressed as a convolution (迴旋積分) of the input *x*[*n*] and the *impulse response h*[*n*]
  - The system can be characterized by the system's *impulse response h*[*n*], which also is a signal sequence
    - If the input *x*[*n*] is impulse $\delta[n]$, the output is *h*[*n*]

Properties of Digital Systems (cont.)

- Linear time-invariant (LTI)
  - Explanation:

$$x[n] = \sum_{k=-\infty}^{\infty} \boxed{x[k]}\,\boxed{\delta[n-k]}$$

scale

Time-shifted unit impulse sequence

$$\Rightarrow T\{x[n]\} = T\left\{\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]\right\}$$

linear

$$= \sum_{k=-\infty}^{\infty} x[k]\,\boxed{T\{\delta[n-k]\}}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\,\boxed{h[n-k]}$$

Time-invariant

$$= x[n] * h[n]$$

convolution

Impulse response

$$\delta[n] \longrightarrow \boxed{\text{Digital System}} \longrightarrow h[n]$$

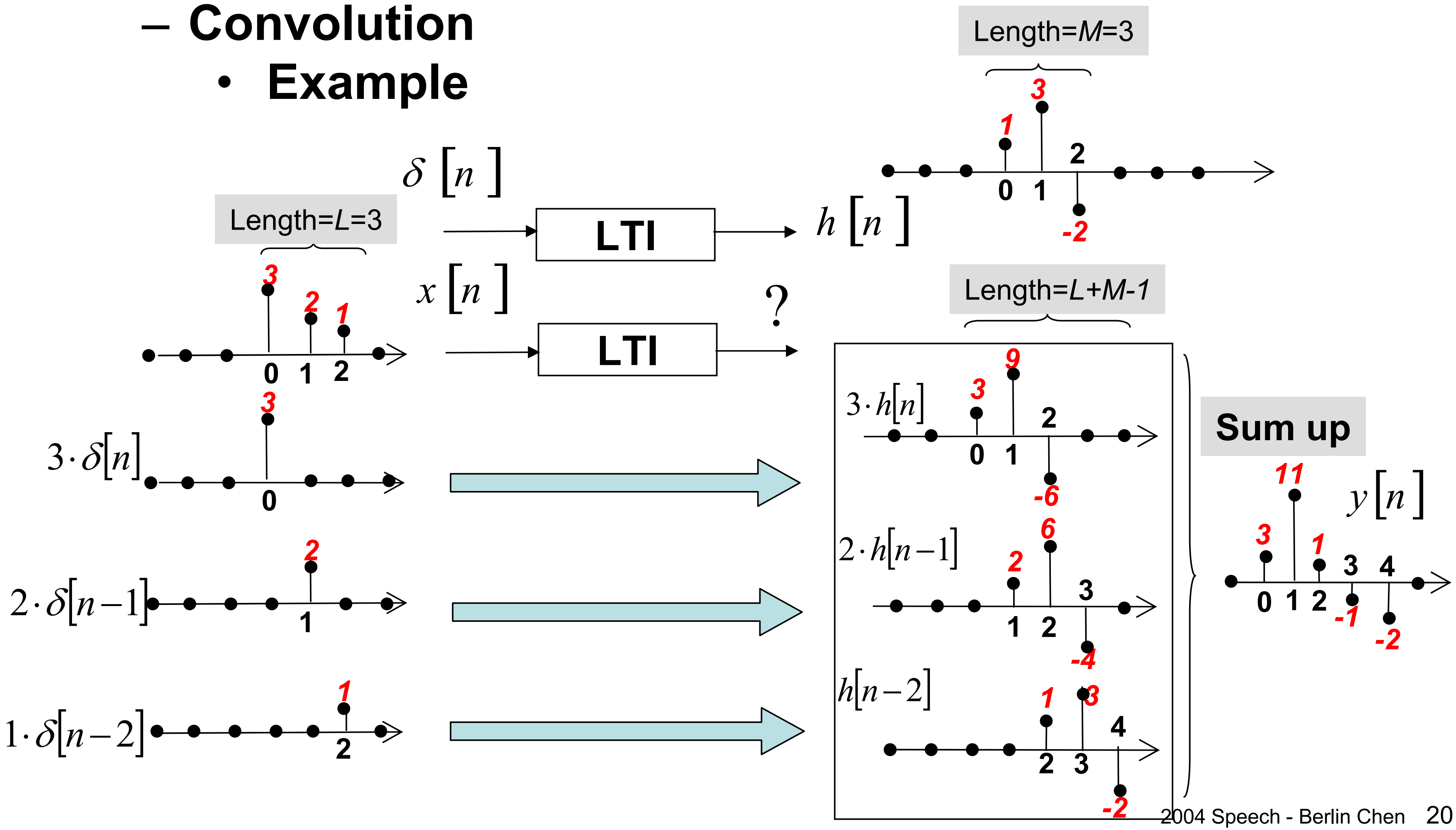
Time invariant

$$\delta[n] \xrightarrow{T} h[n]$$

$$\delta[n-k] \xrightarrow{T} h[n-k]$$

Properties of Digital Systems (cont.)

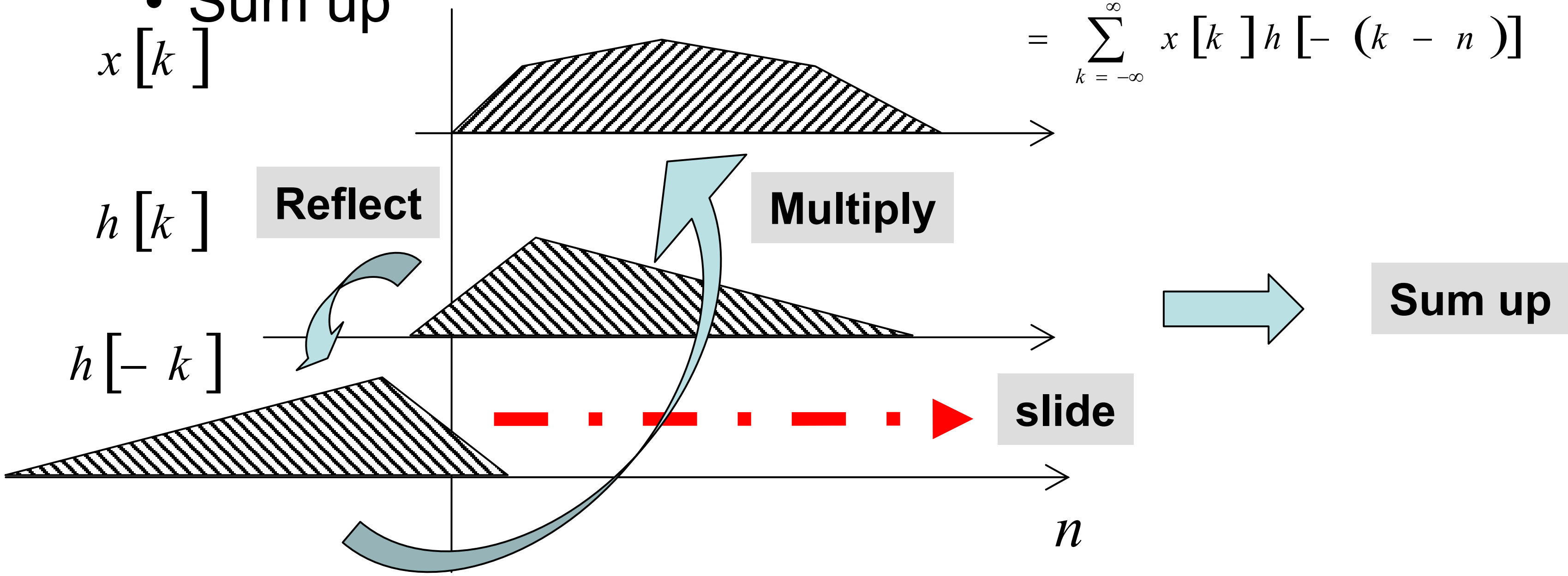
- Linear time-invariant (LTI)
  - **Convolution**
    - **Example**



Properties of Digital Systems (cont.)

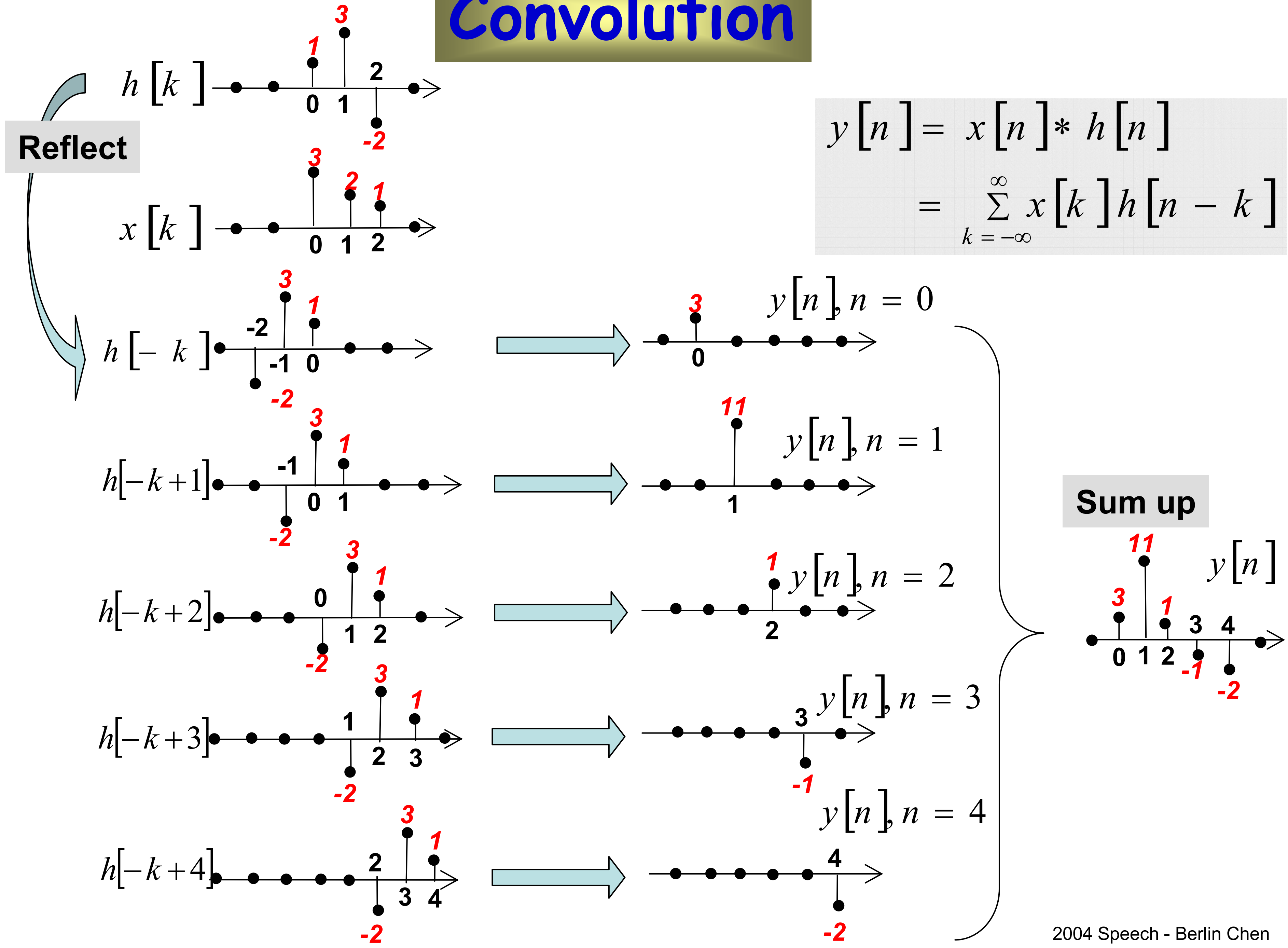
- Linear time-invariant (LTI)
  - Convolution: **Generalization**
    - Reflect $h[k]$ about the origin ($\rightarrow h[-k]$)
    - Slide ($h[-k] \rightarrow h[-k+n]$ or $h[-(k-n)]$), multiply it with $x[k]$
    - Sum up

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$
$$= \sum_{k=-\infty}^{\infty} x[k]h[-(k-n)]$$



Convolution

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

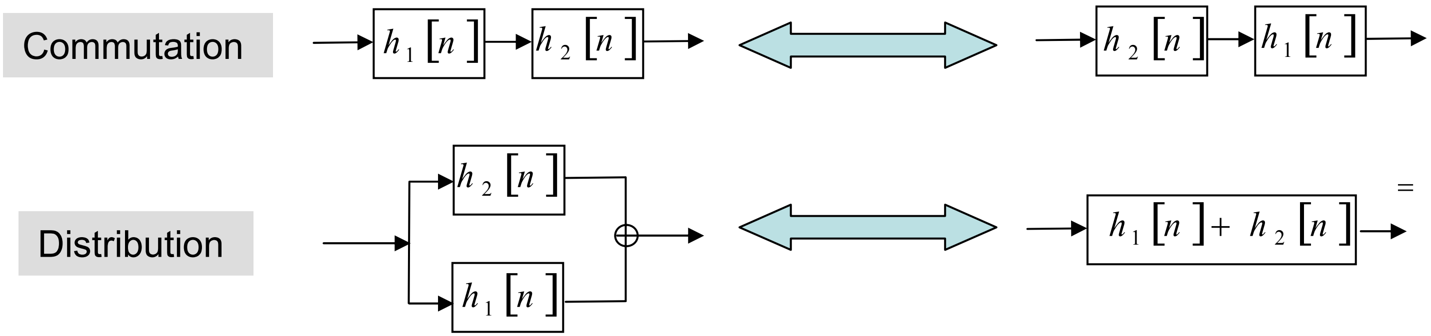


Properties of Digital Systems (cont.)

- Linear time-invariant (LTI)
  - Convolution is commutative and distributive

$$x[n]*h_1[n]*h_2[n]$$
$$= x[n]*h_2[n]*h_1[n]$$

$$x[n]*(h_1[n]+h_2[n])$$
$$= x[n]*h_1[n]+x[n]*h_2[n]$$



- An impulse response has finite duration
  - » **Finite-Impulse Response (FIR)**
- An impulse response has infinite duration
  - » **Infinite-Impulse Response (IIR)**

$$\begin{aligned} y[n] &= x[n]*h[n] \\ &= h[n]*x[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \end{aligned}$$

Properties of Digital Systems (cont.)

- **Bounded Input and Bounded Output (BIBO)**: stable

$$|x[n]| \leq B_x < \infty \quad \forall n$$
$$|y[n]| \leq B_y < \infty \quad \forall n$$

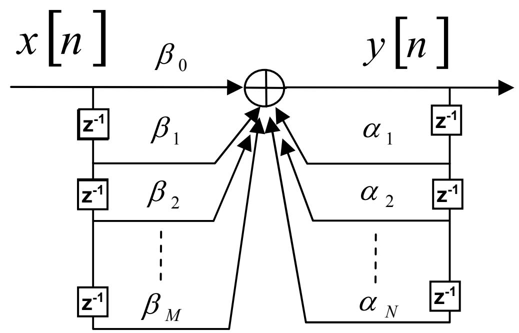
  - **A LTI system** is BIBO if only if $h[n]$ is absolutely summable

$$\sum_{k=-\infty}^{\infty} |h[k]| \leq \infty$$

Properties of Digital Systems (cont.)

- **Causality**
  - A system is "casual" if for every choice of n_0, the output sequence value at indexing $n=n_0$ depends on only the input sequence value for $n \leq n_0$

$$y[n_0] = \sum_{k=1}^{K} \alpha_k y[n_0 - k] + \sum_{k=m}^{M} \beta_k x[n_0 - m]$$



  - Any noncausal FIR can be made causal by adding sufficient long delay

Discrete-Time Fourier Transform (DTFT)

- Frequency Response $H\left(e^{j\omega}\right)$
  - Defined as the discrete-time Fourier Transform of $h[n]$
  - $H\left(e^{j\omega}\right)$ is continuous and is periodic with period= 2π

proportional to two times of the sampling frequency

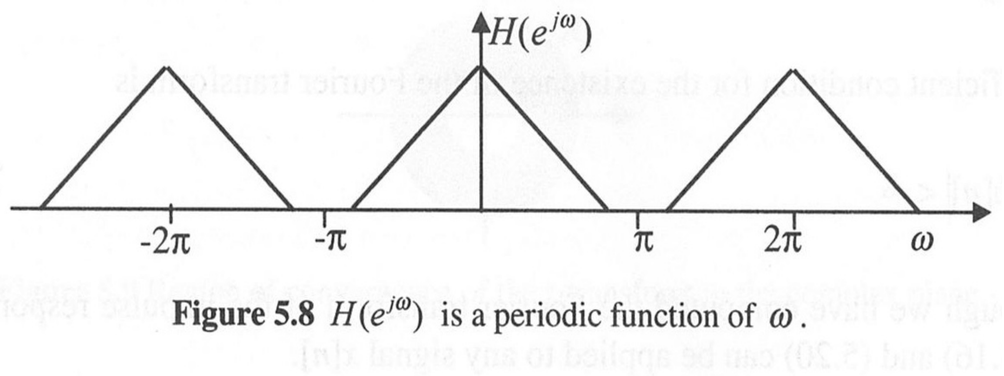


Figure 5.8 $H(e^{j\omega})$ is a periodic function of ω.

  - $H\left(e^{j\omega}\right)$ is a complex function of ω

$$H\left(e^{j\omega}\right) = H_r\left(e^{j\omega}\right) + jH_i\left(e^{j\omega}\right)$$

$$= \underbrace{\left|H\left(e^{j\omega}\right)\right|}_{\text{magnitude}} e^{\underbrace{j\angle H\left(e^{j\omega}\right)}_{\text{phase}}}$$

Discrete-Time Fourier Transform (cont.)

- Representation of Sequences by Fourier Transform

$$H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$ DTFT

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{j\omega}\right) e^{j\omega n} d\omega$$ Inverse DTFT

  - A sufficient condition for the existence of Fourier transform

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$ *absolutely summable*

Fourier transform is invertible:

$$H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{j\omega}\right) e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \boxed{\sum_{m=-\infty}^{\infty} h[m] e^{-j\omega m}} e^{j\omega n} d\omega$$

$$= \sum_{m=-\infty}^{\infty} h[m] \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-m)} d\omega = \sum_{m=-\infty}^{\infty} h[m]\delta[n-m] = h[n]$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-m)} d\omega$$
$$= \frac{1}{j2\pi(n-m)}\left[e^{j\omega(n-m)}\right]_{-\pi}^{\pi}$$
$$= \frac{\sin \pi(n-m)}{\pi(n-m)}$$
$$= \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$
$$= \delta[n-m]$$

Discrete-Time Fourier Transform (cont.)

- Convolution Property

$$H\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

$$y[n]=x[n]*h[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$Y\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}\boxed{\sum_{k=-\infty}^{\infty} x[k]h[n-k]}e^{-j\omega n}$$

$$n'=n-k$$
$$\Rightarrow n=n'+k$$
$$\Rightarrow -n=-n'-k$$

$$=\sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}\left(\sum_{n'=-\infty}^{\infty} h[n']e^{-j\omega n'}\right)$$

$$=X\left(e^{j\omega}\right)H\left(e^{j\omega}\right)$$

$$\therefore\ x[n]*h[n]\Leftrightarrow X\left(e^{j\omega}\right)H\left(e^{j\omega}\right)$$

$$Y\left(e^{j\omega}\right)=X\left(e^{j\omega}\right)H\left(e^{j\omega}\right)$$
$$\Rightarrow \left|Y\left(e^{j\omega}\right)\right|=\left|X\left(e^{j\omega}\right)\right|\left|H\left(e^{j\omega}\right)\right|$$
$$\Rightarrow \angle Y\left(e^{j\omega}\right)=\angle X\left(e^{j\omega}\right)+\angle H\left(e^{j\omega}\right)$$

Discrete-Time Fourier Transform (cont.)

- Parseval's Theorem

power spectrum

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \boxed{\left|X\left(e^{j\omega}\right)\right|^2} d\omega$$

The total energy of a signal can be given in either the time or frequency domain.

– Define the autocorrelation of signal $x[n]$

$$R_{xx}[n] = \sum_{m=-\infty}^{\infty} x[m+n]x^*[m]$$

$l = m+n$
$\Rightarrow m = l-n = -(n-l)$

$$\Leftrightarrow \quad = \sum_{l=-\infty}^{\infty} x[l]x^*[-(n-l)] = x[n] * x^*[-n]$$

$$S_{xx}(\omega) = X(\omega)X^*(\omega) = |X(\omega)|^2$$

$$R_{xx}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xx}(\omega)e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 e^{j\omega n}d\omega$$

Set $n = 0$

$$R_{xx}[0] = \sum_{m=-\infty}^{\infty} x[m]x^*[m] = \sum_{m=-\infty}^{\infty} |x[m]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

Discrete-Time Fourier Transform (cont.)

Property	Signal	Fourier Transform
Linearity	$ax_1[n]+bx_2[n]$	$aX_1(e^{j\omega})+bX_2(e^{j\omega})$
Symmetry	$x[-n]$	$X(e^{-j\omega})$
	$x^*[n]$	$X^*(e^{-j\omega})$
	$x^*[-n]$	$X^*(e^{j\omega})$
	$x[n]$ real	$X(e^{j\omega})$ is *Hermitian* $X(e^{-j\omega})=X^*(e^{j\omega})$ $\left\|X(e^{j\omega})\right\|$ is even[6] $\mathrm{Re}\{X(e^{j\omega})\}$ is even $\arg\left\{X(e^{j\omega})\right\}$ is odd[7] $\mathrm{Im}\left\{X(e^{j\omega})\right\}$ is odd
	$\mathrm{Even}\{x[n]\}$	$\mathrm{Re}\{X(e^{j\omega})\}$
	$\mathrm{Odd}\{x[n]\}$	$j\,\mathrm{Im}\{X(e^{j\omega})\}$
Time-shifting	$x[n-n_0]$	$X(e^{j\omega})e^{-j\omega n_0}$
Modulation	$x[n]e^{j\omega_0 n}$	$X(e^{j(\omega-\omega_0)})$
	$x[n]z_0^n$	
Convolution	$x[n]*h[n]$	$X(e^{j\omega})H(e^{j\omega})$
	$x[n]y[n]$	$\frac{1}{2\pi}X(e^{j\omega})*Y(e^{j\omega})$
Parseval's Theorem	$R_{xx}[n]=\sum_{m=-\infty}^{\infty}x[m+n]x^*[m]$	$S_{xx}(\omega)=\left\|X(\omega)\right\|^2$

Z-Transform

- *z*-transform is a generalization of (Discrete-Time) Fourier transform

$$h[n] \longrightarrow H\left(e^{j\omega}\right)$$

$$h[n] \longrightarrow H(z)$$

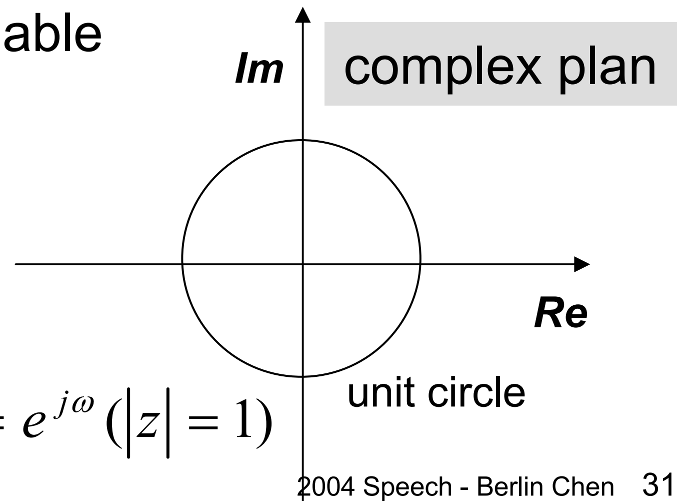
– z-transform of $h[n]$ is defined as

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

- Where $z = re^{j\omega}$, a complex-variable
- For Fourier transform

$$H\left(e^{j\omega}\right) = H(z)\Big|_{z=e^{j\omega}}$$

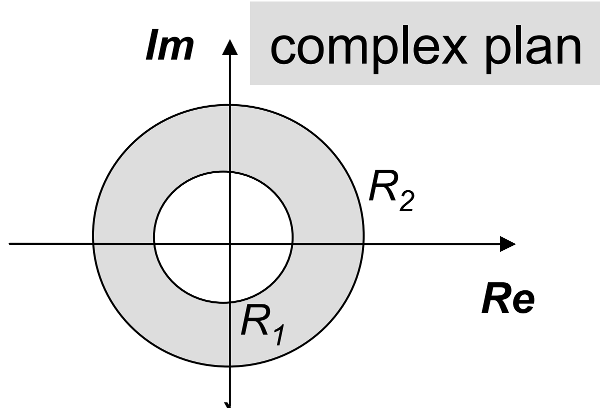
– z-transform evaluated on the unit circle



$$z = e^{j\omega} \; (|z| = 1)$$

Z-Transform (cont.)

- Fourier transform vs. *z*-transform
  - Fourier transform used to plot the frequency response of a filter
  - z-transform used to analyze more general filter characteristics, e.g. stability



- ROC (Region of Converge)
  - Is the set of *z* for which *z*-transform exists (converges)

$$\sum_{n=-\infty}^{\infty} |h[n]||z|^{-n} < \infty$$

absolutely summable

  - In general, ROC is a **ring-shaped region** and the Fourier transform exists if ROC includes the unit circle

Z-Transform (cont.)

$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= h[n] * x[n] \\
&= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]
\end{aligned}
$$

- An LTI system is defined to be ***causal***, if its impulse response is a causal signal, i.e.

$$h[n] = 0 \quad \text{for} \quad n < 0$$

Right-sided sequence

  – Similarly, ***anti-causal*** can be defined as

$$h[n] = 0 \quad \text{for} \quad n > 0$$

Left-sided sequence

- An LTI system is defined to be ***stable***, if for every bounded input it produces a bounded output
  – Necessary condition:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

    - That is Fourier transform exists, and therefore z-transform include the unit circle in its region of converge

Z-Transform (cont.)

- **Right-Sided Sequence**
  - E.g., the exponential signal

1. $h_1[n] = a^n u[n]$, where $u[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$

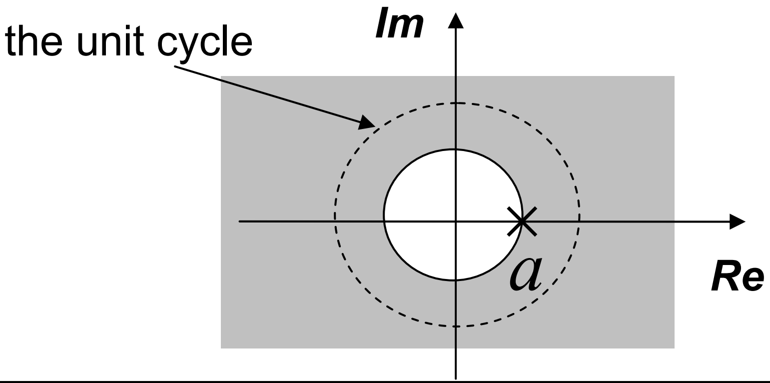
$$H_1(z) = \sum_{n=-\infty}^{\infty} a^n z^{-n} = \sum_{n=-\infty}^{\infty} \left(az^{-1}\right)^n = \frac{1}{1 - az^{-1}}$$

have a pole at $z = a$

(Pole: z-transform goes to infinity)

If $\left|az^{-1}\right| < 1$

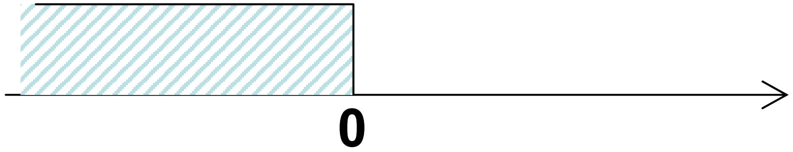
$$\therefore\ ROC_1 \text{ is } |z| > |a|$$



Fourier transform of $h_1[n]$ exists if $|a| < 1$

Z-Transform (cont.)

- **Left-Sided Sequence**
  - E.g.



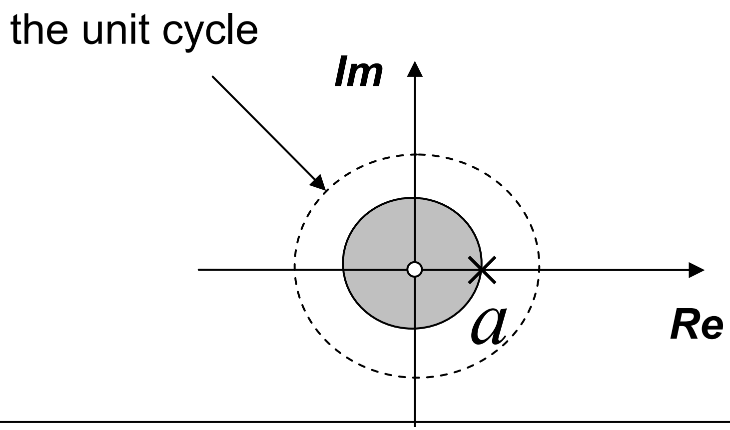
2. $h_2[n] = -a^n u[-n-1]$

$$H_2(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

If $\left|a^{-1}z\right| < 1$

$$= -\sum_{n=1}^{\infty} a^{-n} z^{n} = 1 - \sum_{n=0}^{\infty} \left(a^{-1}z\right)^n = 1 - \frac{1}{1-a^{-1}z} = -\frac{-a^{-1}z}{1-a^{-1}z} = \frac{1}{1-az^{-1}}$$

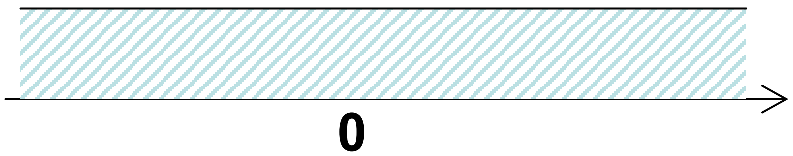
$$\therefore\ ROC_2 \text{ is } |z| < |a|$$



when $|a| < 1$, the Fourier transform of $h_2[n]$ doesn't exist, because $h_2[n]$ will go exponentially as $n \to -\infty$

Z-Transform (cont.)

- **Two-Sided Sequence**



  - E.g.

$$3. \quad h_3[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

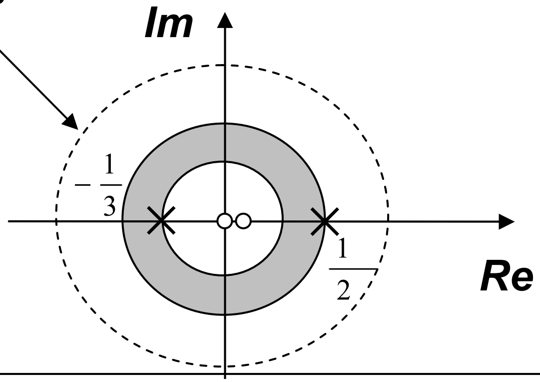
$$\left(-\frac{1}{3}\right)^n u[n] \xleftrightarrow{z} \frac{1}{1+\frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$$

$$-\left(\frac{1}{2}\right)^n u[-n-1] \xleftrightarrow{z} \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| < \frac{1}{2}$$

$$H_3(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

$$\therefore \; ROC_3 \text{ is } |z| < \left|\frac{1}{2}\right| \text{ and } |z| > \left|\frac{1}{3}\right|$$

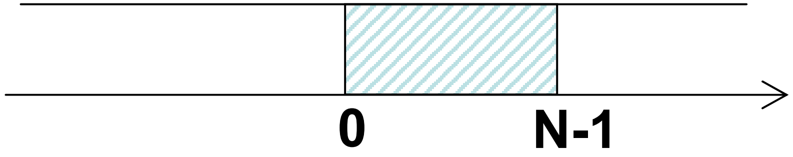
the unit cycle



Fourier transform of $h_3[n]$ doesn't exist, because ROC_3 doesn't include the unit circle

Z-Transform (cont.)

- **Finite-length Sequence**



  - E.g.

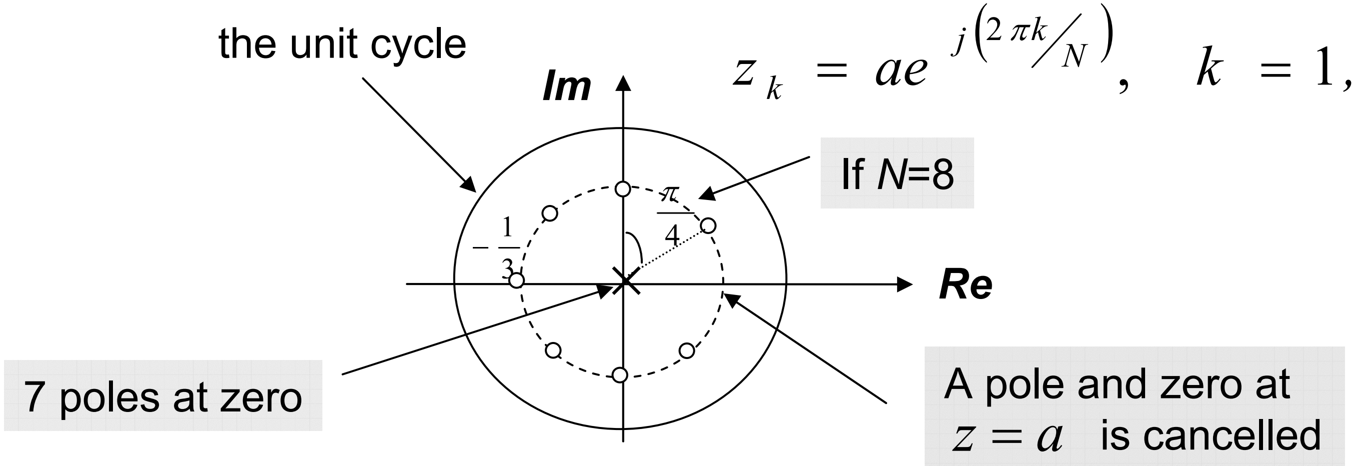
3. $$h_4[n]=\begin{cases} a^n, & 0\le n\le N-1 \\ 0, & \text{others} \end{cases}$$

$$z^{N-1}+az^{N-2}+a^2z^{N-3}+.....+a^{N-1}$$

$$H_4(z)=\sum_{n=0}^{N-1}a^n z^{-n}=\sum_{n=0}^{N-1}\left(az^{-1}\right)^n=\frac{1-\left(az^{-1}\right)^N}{1-az^{-1}}=\frac{1}{z^{N-1}}\frac{\cancel{z^N-a^N}}{\cancel{z-a}}$$

$$\therefore ROC_4 \text{ is entire } z\text{-plane except } z=0$$

$$z_k = ae^{j\left(2\pi k/N\right)},\quad k=1,\ldots,N-1$$



Z-Transform (cont.)

- **Properties of *z*-transform**

  1. If $h[n]$ is right-sided sequence, i.e. $h[n] = 0,\ n \le n_1$ and if *ROC* is the exterior of some circle, the **all finite** z for which $|z| > r_0$ will be in *ROC*
     - If $n_1 \ge 0$,*ROC* will include $z = \infty$

A causal sequence is right-sided with $n_1 \ge 0$
$\therefore$ ROC is the exterior of circle including $z = \infty$

  2. If $h[n]$ is left-sided sequence, i.e. $h[n] = 0,\ n \ge n_2$, the *ROC* is the interior of some circle, $z = 0$
     - If $n_2 < 0$,*ROC* will include
  3. If $h[n]$ is two-sided sequence, the *ROC* is a **ring**
  4. The ROC can't contain any poles

Summary of the Fourier and *z*-transforms

Table 5.5 Properties of the Fourier and *z*-transforms.

Property	Signal	Fourier Transform	*z*-Transform
Linearity	$ax_1[n]+bx_2[n]$	$aX_1(e^{j\omega})+bX_2(e^{j\omega})$	$aX_1(z)+bX_2(z)$
Symmetry	$x[-n]$	$X(e^{-j\omega})$	$X(z^{-1})$
	$x^*[n]$	$X^*(e^{-j\omega})$	$X^*(z^*)$
	$x^*[-n]$	$X^*(e^{j\omega})$	$X^*(1/z^*)$
	$x[n]$ real	$X(e^{j\omega})$ is *Hermitian* $X(e^{-j\omega})=X^*(e^{j\omega})$ $\lvert X(e^{j\omega})\rvert$ is even[6] $\mathrm{Re}\{X(e^{j\omega})\}$ is even $\arg\{X(e^{j\omega})\}$ is odd[7] $\mathrm{Im}\{X(e^{j\omega})\}$ is odd	$X(z^*)=X^*(z)$
	$\mathrm{Even}\{x[n]\}$	$\mathrm{Re}\{X(e^{j\omega})\}$	
	$\mathrm{Odd}\{x[n]\}$	$j\,\mathrm{Im}\{X(e^{j\omega})\}$	
Time-shifting	$x[n-n_0]$	$X(e^{j\omega})e^{-j\omega n_0}$	$X(z)z^{-n_0}$
Modulation	$x[n]e^{j\omega_0 n}$	$X(e^{j(\omega-\omega_0)})$	$X(e^{-j\omega_0}z)$
	$x[n]z_0^n$		$X(z/z_0)$
Convolution	$x[n]*h[n]$	$X(e^{j\omega})H(e^{j\omega})$	$X(z)H(z)$
	$x[n]y[n]$	$\frac{1}{2\pi}X(e^{j\omega})*Y(e^{j\omega})$	
Parseval's Theorem	$R_{xx}[n]=\sum_{m=-\infty}^{\infty}x[m+n]x^*[m]$	$S_{xx}(\omega)=\lvert X(\omega)\rvert^2$	$X(z)X^*(1/z^*)$

LTI Systems in the Frequency Domain

- **Example 1**: A complex exponential sequence $x[n] = e^{j\omega n}$
  - System impulse response $h[n]$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)}$$

$$= e^{j\omega n} \boxed{\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}}$$

$$= H\left(e^{j\omega}\right) e^{j\omega n}$$

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= h[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} x[k] h[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k] x[n-k] \end{aligned}$$

$H\left(e^{j\omega}\right)$: the Fourier transform of the system impulse response. It is often referred to as the system frequency response.

  - Therefore, a complex exponential input to an LTI system results in the same complex exponential at the output, but modified by $H\left(e^{j\omega}\right)$
    - The complex exponential is an eigenfunction of an LTI system, and $H\left(e^{j\omega}\right)$ is the associated eigenvalue

$$T\{x[n]\} = H\left(e^{j\omega}\right) x[n]$$

LTI Systems in the Frequency Domain (cont.)

- **Example 2**: A sinusoidal sequence

$$x[n] = A\cos(w_0 n + \phi)$$

$$e^{j\theta} = \cos\theta + i\sin\theta$$
$$e^{-j\theta} = \cos\theta - i\sin\theta$$
$$\Rightarrow \cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$

$$x[n] = A\cos(\omega_0 n + \phi)$$
$$= \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$$

- System impulse response $h[n]$

$$z = x + jy \qquad \Rightarrow e^{j\omega} = \cos\omega + j\sin\omega$$
$$z^* = x - jy \qquad \Rightarrow \left(e^{j\omega}\right)^* = \cos\omega - j\sin\omega$$
$$e^{-j\omega} = \cos(-\omega) + j\sin(-\omega)$$
$$= \cos\omega - j\sin\omega$$

$$H\left(e^{-j\omega_0}\right) = H^*\left(e^{j\omega_0}\right)$$
$$H^*\left(e^{j\omega_0}\right) = \left|H\left(e^{j\omega_0}\right)\right| e^{-j\angle H\left(e^{j\omega_0}\right)}$$

$$y[n] = H\left(e^{j\omega_0}\right)\frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \boxed{H\left(e^{-j\omega_0}\right)}\frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$$
$$= \frac{A}{2}\left[\boxed{H\left(e^{j\omega_0}\right)}e^{j(\omega_0 n + \phi)} + \boxed{H^*\left(e^{j\omega_0}\right)}e^{-j(\omega_0 n + \phi)}\right]$$
$$= \frac{A}{2}\left[\boxed{\left|H\left(e^{j\omega_0}\right)\right| e^{j\angle H\left(e^{j\omega_0}\right)}}e^{j(\omega_0 n + \phi)} + \boxed{\left|H\left(e^{j\omega_0}\right)\right| e^{-j\angle H\left(e^{j\omega_0}\right)}}e^{-j(\omega_0 n + \phi)}\right]$$
$$= A\left|H\left(e^{j\omega_0}\right)\right|\cos\left[\omega_0 n + \phi + \angle H\left(e^{j\omega_0}\right)\right]$$

LTI Systems in the Frequency Domain (cont.)

- **Example 3**: A sum of sinusoidal sequences

$$x[n] = \sum_{k=1}^{K} A_k \cos(\omega_k n + \phi_k)$$

$$y[n] = \sum_{k=1}^{K} A_k \left|H\left(e^{j\omega_k}\right)\right| \cos\left[\omega_k n + \phi_k + \angle H\left(e^{j\omega_k}\right)\right]$$

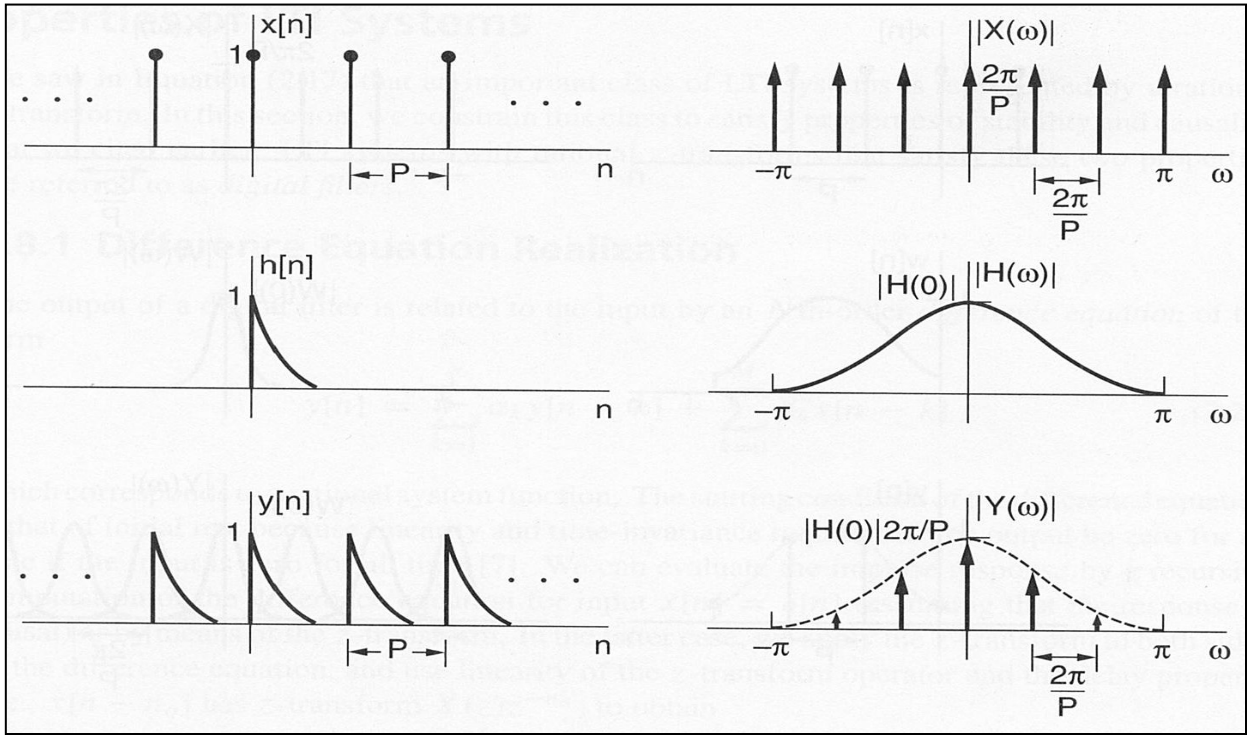
  – A similar expression is obtained for an input consisting of a sum of complex exponentials

LTI Systems in the Frequency Domain (cont.)

- **Example 4: Convolution Theorem** $x[n]*h[n] \Leftrightarrow X(e^{j\omega})H(e^{j\omega})$

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-kP] \quad \xrightarrow{\textbf{DTFT}} \quad X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{P}\delta\left[\left(\omega - \frac{2\pi}{P}k\right)\right]$$

$$h[n] = \sum_{k=-\infty}^{\infty} a^n u[n],\ |a|<1 \quad \xrightarrow{\textbf{DTFT}} \quad H(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$$



$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$= \frac{1}{1-ae^{-j\omega}} \sum_{k=-\infty}^{\infty} \frac{2\pi}{P}\delta\left[\left(\omega - \frac{2\pi}{P}k\right)\right]$$

$$= \frac{2\pi}{P} \sum_{k=-\infty}^{\infty} \frac{1}{1-ae^{-j\frac{2\pi}{P}k}}\delta\left[\left(\omega - \frac{2\pi}{P}k\right)\right]$$

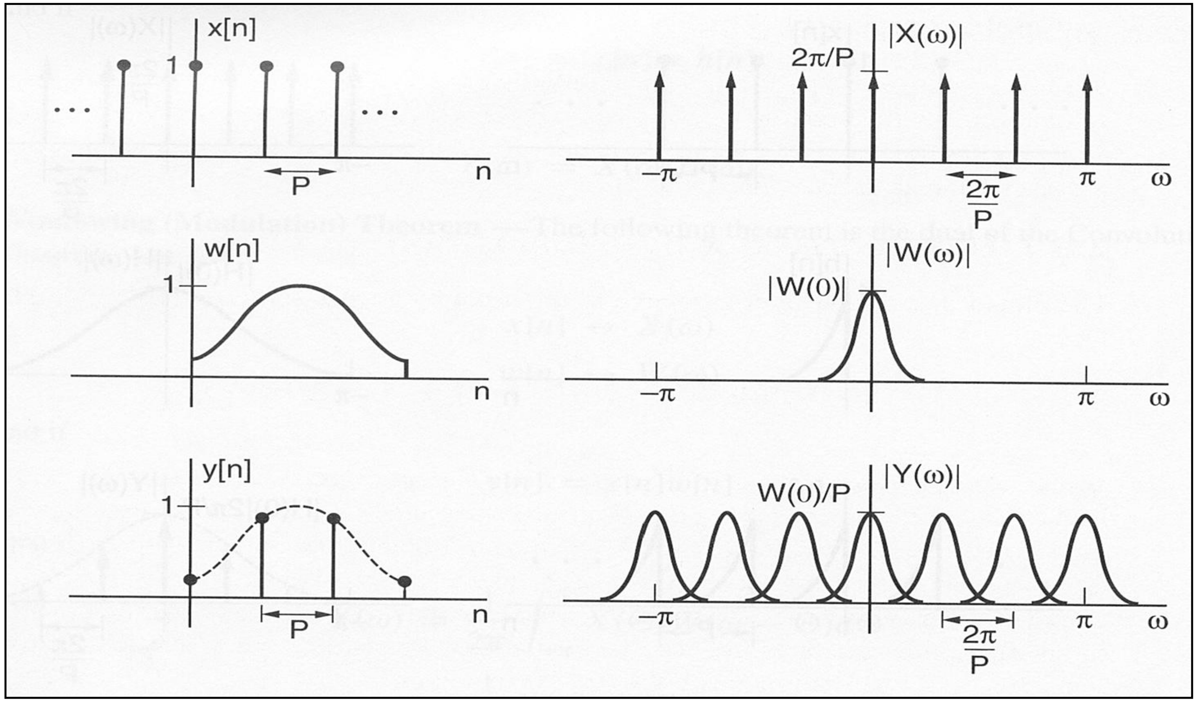
LTI Systems in the Frequency Domain (cont.)

- **Example 5**: **Windowing Theorem** $x[n]w[n] \Leftrightarrow \frac{1}{2\pi} W(e^{j\omega}) * X(e^{j\omega})$

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kP]$$

$$w[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), & n = 0,1,......,N-1 \\ 0 & \text{otherwise} \end{cases}$$

Hamming window



$$Y(e^{j\omega}) = \frac{1}{2\pi} W(e^{j\omega}) * X(e^{j\omega})$$

$$= \frac{1}{2\pi} W(e^{j\omega}) * \sum_{k=-\infty}^{\infty} \frac{2\pi}{P} \delta\left(\omega - \frac{2\pi}{P}k\right)$$

$$= \frac{1}{P} \sum_{k=-\infty}^{\infty} \left\{ W(e^{j\omega}) * \delta\left(\omega - \frac{2\pi}{P}k\right) \right\}$$

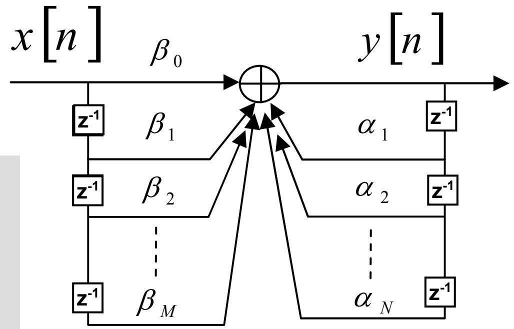
$$= \frac{1}{P} \sum_{k=-\infty}^{\infty} \left\{ \sum_{m=-\infty}^{m=\infty} W(e^{jm}) \delta\left(\omega - \frac{2\pi}{P}k - m\right) \right\}$$

$$= \frac{1}{P} \sum_{k=-\infty}^{\infty} W\left(e^{j\left(\omega - \frac{2\pi}{P}k\right)}\right)$$

Difference Equation Realization for a Digital Filter

- The relation between the output and input of a digital filter can be expressed by

$$y[n]=\sum_{k=1}^{N}\alpha_k\, y[n-k]+\sum_{k=0}^{M}\beta_k\, x[n-k]$$



linearity and delay properties

delay property

$$x[n]\to X(z)$$

$$x[n-n_0]\to X(z)z^{-n_0}$$

$$Y(z)=\sum_{k=1}^{N}\alpha_k\, Y(z)z^{-k}+\sum_{k=0}^{M}\beta_k\, X(z)z^{-k}$$

A rational transfer function

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_{k=0}^{M}\beta_k z^{-k}}{1-\sum_{k=1}^{N}\alpha_k z^{-k}}$$

Causal:
Rightsided, the ROC outside the outmost pole
Stable:
The ROC includes the unit circle
Causal and Stable:
all poles must fall inside the unit circle (not including zeros)

Difference Equation Realization for a Digital Filter (cont.)

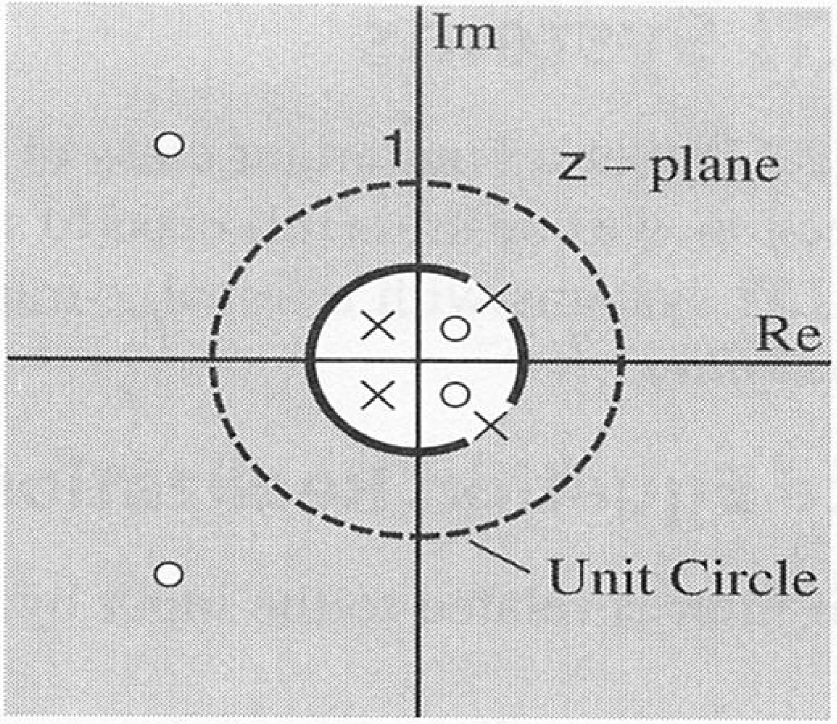


Figure 2.8 Pole-zero configuration for a causal and stable discrete-time system.

Magnitude-Phase Relationship

- **Minimum phase system:**
  - The z-transform of a system impulse response sequence (a rational transfer function) has all zeros as well as poles inside the unit cycle
  - Poles and zeros called “minimum phase components”
  - **Maximum phase**: all zeros (or poles) outside the unit cycle
- **All-pass system:**
  - Consist a cascade of factor of the form

$$\left[\frac{1\text{-}a^{*}z}{1-az^{-1}}\right]^{\pm 1}$$

  - Characterized by a frequency response with unit (or flat) magnitude for all frequencies $\left|\frac{1\text{-}a^{*}z}{1-az^{-1}}\right| = 1$

Poles and zeros occur at conjugate reciprocal locations

Magnitude-Phase Relationship (cont.)

- Any digital filter can be represented by the cascade of a minimum-phase system and an all-pass system

$$H(z) = H_{\min}(z) H_{ap}(z)$$

Suppose that $H(z)$ has only one zero $1/a^*$ $(|a| < 1)$ outside the unit circle. $H(z)$ can be expressed as:

$$H(z) = H_1(z)\left(1 - a^* z\right) \quad (H_1(z) \text{ is a minimum phase filter})$$

$$= H_1(z)\left(1 - a z^{-1}\right)\frac{\left(1 - a^* z\right)}{\left(1 - a z^{-1}\right)}$$

where:

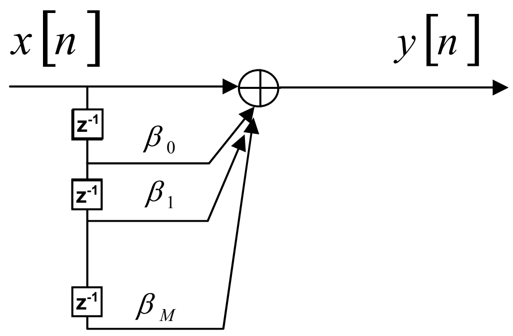
$H_1(z)\left(1 - a z^{-1}\right)$ is also a minimum phase filter.

$\frac{\left(1 - a^* z\right)}{\left(1 - a z^{-1}\right)}$ is a all-pass filter.

FIR Filters

- FIR (Finite Impulse Response)
  - The impulse response of an FIR filter has finite duration
  - Have no denominator in the rational function
    - No feedback in the difference equation

$H(z)$



$$y[n] = \sum_{r=0}^{M} \beta_r x[n-r]$$

$$h[n] = \begin{cases} \beta_n, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{M} \beta_k z^{-k}$$

  - Can be implemented with simple a train of delay, multiple, and add operations

First-Order FIR Filters

- A special case of FIR filters

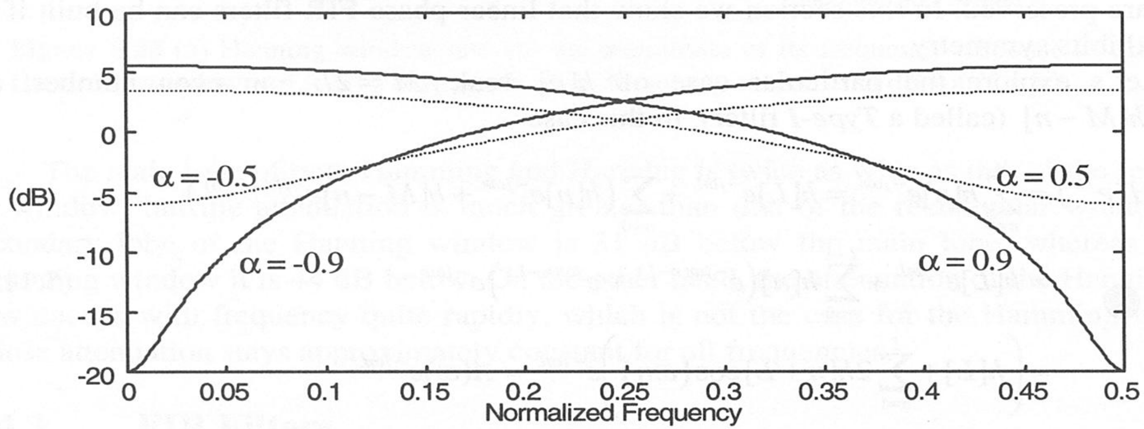
$$y[n] = x[n] + \alpha x[n-1] \quad \Longleftrightarrow \quad H(z) = 1 + \alpha z^{-1}$$

$$H\left(e^{j\omega}\right) = 1 + \alpha e^{-j\omega}$$

$$\left|H\left(e^{j\omega}\right)\right|^2 = \left|1 + \alpha(\cos\omega - j\sin\omega)\right|^2$$

$$= (1 + \alpha\cos\omega)^2 + (\alpha\sin\omega)^2 = 1 + 2\alpha\cos\omega$$

$$\theta\left(e^{j\omega}\right) = -\arctan\left(\frac{\alpha\sin\omega}{1 + \alpha\cos\omega}\right)$$



$$10\log\left|H\left(e^{j\omega}\right)\right|^2$$

Figure 5.21 Frequency response of the first order FIR filter for various values of α.

Discrete Fourier Transform (DFT)

- The Fourier transform of a discrete-time sequence is a continuous function of frequency
  - We need to sample the Fourier transform finely enough to be able to recover the sequence
  - For a sequence of finite length *N*, sampling yields the new transform referred to as *discrete Fourier transform* (DFT)

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}, \quad 0 \le n \le N-1$$

DFT, **Analysis**

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}, \quad 0 \le n \le N-1$$

Inverse DFT, **Synthesis**

Discrete Fourier Transform (cont.)

$$\forall\ 0 \le k \le N-1$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}, \quad 0 \le n \le N-1$$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-j\frac{2\pi}{N}1\cdot 1} & \cdots & e^{-j\frac{2\pi}{N}1\cdot(N-1)} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & e^{-j\frac{2\pi}{N}(N-1)\cdot 1} & \cdots & e^{-j\frac{2\pi}{N}(N-1)\cdot(N-1)} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} = \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$$

Discrete Fourier Transform (cont.)

- Orthogonality of Complex Exponentials

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-r)n} = \begin{cases} 1, & \text{if } k\text{-}r = mN \\ 0, & \text{otherwise} \end{cases}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn}$$

$$\Rightarrow \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}rn} = \sum_{n=0}^{N-1}\frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}(k-r)n}$$

$$= \sum_{k=0}^{N-1} X[k]\left[\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-r)n}\right]$$

$$= X[r]$$

$$\Rightarrow X[k] = \sum_{r=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$$

$$X[k] = X[r+mN] = X[r]$$

Discrete Fourier Transform (DFT)

- Parseval's theorem

$$\sum_{n=0}^{N-1} \left| x[n] \right|^2 = \frac{1}{N} \sum_{k=0}^{N-1} \left| X(k) \right|^2$$

Energy density